

BIJECTIVE PROJECTIONS ON PARABOLIC QUOTIENTS OF AFFINE WEYL GROUPS

ELIZABETH BEAZLEY, MARGARET NICHOLS, MIN HAE PARK, XIAOLIN SHI, AND ALEXANDER YOUNG

ABSTRACT. Affine Weyl groups and their parabolic quotients are used extensively as indexing sets for objects in combinatorics, representation theory, algebraic geometry, and number theory. Moreover, in the classical Lie types we can conveniently realize the elements of these quotients via intuitive geometric and combinatorial models such as abaci, alcoves, coroot lattice points, core partitions, and bounded partitions. In [1] Berg, Jones, and Vazirani described a bijection between n -cores with first part equal to k and $(n-1)$ -cores with first part less than or equal to k , and they interpret this bijection in terms of these other combinatorial models for the quotient of the affine symmetric group by the finite symmetric group. In this paper we discuss how to generalize the bijection of Berg-Jones-Vazirani to parabolic quotients of affine Weyl groups in other classical Lie types. We develop techniques using the associated affine hyperplane arrangement to interpret this bijection geometrically as a projection of alcoves onto the hyperplane containing their coroot lattice points. We are thereby able to analyze this bijective projection in the language of various additional combinatorial models developed by Hanusa and Jones in [10], such as abaci, core partitions, bounded partitions, and canonical reduced expressions in the Coxeter group.

1. INTRODUCTION

Core partitions are families of Young diagrams which initially arose in the context of studying the representation theory of the symmetric group over a finite field, but cores now appear as indexing sets for many objects in combinatorics, representation theory, algebraic geometry, and number theory. Garvan, Kim, and Stanton combinatorially proved Ramanujan's congruences for the partition function using statistics called cranks, which are closely related to core partitions [8]. In modular representation theory, cores index blocks in the decomposition of the group algebra; see [12]. Also referred to as n -restricted partitions, n -cores correspond to extremal vectors in a highest weight crystal for $\widehat{\mathfrak{sl}}_n$; see [19]. In algebraic geometry, core partitions are connected to expansions of the k -Schur functions of Lapointe, Lascoux, and Morse [15], which were proved to represent the Schubert basis in the homology of the affine Grassmannian by Lam [14], and cores are also related to rational smoothness of Schubert varieties inside the affine Grassmannian as shown by Billey and Mitchell [3]. Exciting new connections to automorphic forms are also emerging in the context of studying the local structure of multiple Dirichlet series in analytic number theory.

In work related to the study of irreducible Specht modules over the Hecke algebra of the symmetric group [2], Berg and Vazirani prove that there is a bijection between the set \mathcal{C}_n^k of n -cores with first part equal to k and the set $\mathcal{C}_{n-1}^{\leq k}$ of $(n-1)$ -cores with first part less than or equal to k . Because of the wide array of connections among core partitions and other areas of mathematics, many additional combinatorial models for core partitions have been developed. For example, n -cores also index minimal length coset representatives in the quotient \hat{S}_n/S_n of the affine symmetric group by the finite symmetric group. There are also interpretations in terms of abacus diagrams, root lattice points, bounded partitions, and certain alcoves in the affine hyperplane arrangement corresponding to the affine symmetric group. Connections among these models play crucial roles in

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various areas of mathematics; for example, the connection between cores and bounded partitions was fundamental in the development of the k -Schur functions. In [1], Berg, Jones, and Vazirani interpret the equipotence of \mathcal{C}_n^k and $\mathcal{C}_{n-1}^{\leq k}$ geometrically in terms of the alcove model for $\tilde{\mathcal{S}}_n/\mathcal{S}_n$, thereby obtaining several additional combinatorial descriptions for this bijection.

Our main theorem is a generalization of the results in [1] to Lie type C . In type C , this parabolic quotient is known to be in bijection with the set of lecture hall partitions introduced by Bousquet-Mélou and Eriksson [6] and certain mirrored \mathbb{Z} -permutations defined by Eriksson in his thesis [7]. Hanusa and Jones additionally define bijections from the quotient \tilde{C}_n/C_n to symmetric core partitions, abacus diagrams, bounded partitions, and canonical reduced expressions in the Coxeter group in [10]. There is also a classical geometric connection to certain hyperplane arrangements through the language of root systems; see [5] and [11].

The crucial ingredient in the proof of many of the results contained in this paper is the ability to provide a geometric interpretation in terms of alcove walks in the affine hyperplane arrangement. These piecewise linear paths in the real span of the weight lattice were introduced by Littelmann, who calls them Lakshmibai-Seshadri paths [13], in order to prove a Littlewood-Richardson rule for decomposing the tensor product of two simple highest weight modules of a complex symmetrizable Kac-Moody algebra into its irreducible components [18]. Alcove walks now arise throughout the literature in representation theory and algebraic geometry, and they often seem in many instances to provide the most natural framework for type-free generalizations of results which had previously only been known in type A . For example, Schwer provides a formula for the Hall-Littlewood polynomials of arbitrary type in terms of alcove walks [23], and this formula was generalized by Ram and Yip to Macdonald polynomials using similar language [22]. There is also an explicit correspondence between alcove walks and saturated chains in strong Bruhat order on the affine Weyl group, which gives rise to type-free applications in equivariant K -theory of flag varieties [17] and the uniform construction of tensor products of certain Kirillov-Reshetikhin crystals [16].

1.1. Summary of the main results. We start by defining a map Φ_n on elements of the parabolic quotient \tilde{C}_n/C_n . The map Φ_n acts on symmetric $(2n)$ -cores, which index the minimal length coset representatives of \tilde{C}_n/C_n , as proved in [10]. Given a symmetric $(2n)$ -core, the map Φ_n acts as follows: first, label the boxes (i, j) of the $(2n)$ -core by the residues $j - i \pmod{2n}$. Then, delete all the rows that end with the same residue as the first row. Finally, delete all the columns that end with the same residue as the first column. It can be shown that the image of Φ_n is a set of $(2n - 2)$ -cores, which correspond to minimal length coset representatives of \tilde{C}_{n-1}/C_{n-1} .

Theorem A (Theorem 5.8). *The map Φ_n^k given by restricting Φ_n to \mathcal{S}_{2n}^k , the set of symmetric $(2n)$ -cores with first part equal to k , becomes a bijection onto its image $\mathcal{S}_{2n-2}^{\leq k - \lceil \frac{k}{n} \rceil}$, the set of symmetric $(2n - 2)$ -cores with first part at most $k - \lceil \frac{k}{n} \rceil$.*

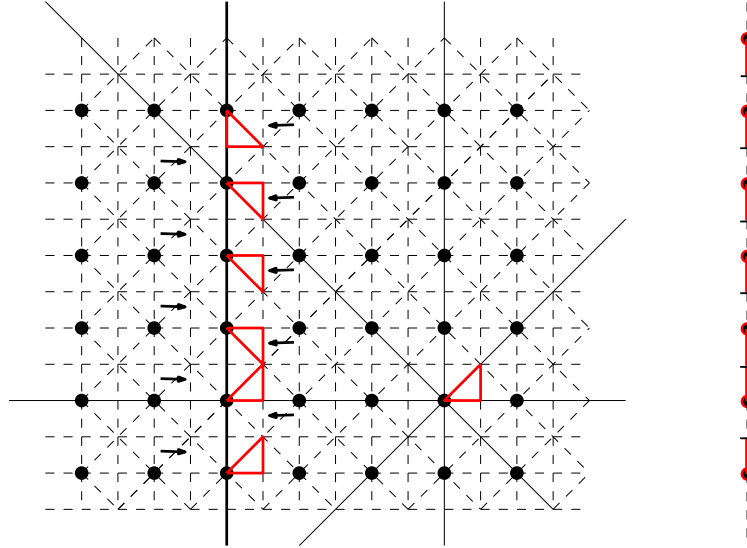
The proof of Theorem A uses the bijection between symmetric $(2n)$ -cores and balanced flush abacus diagrams with $(2n)$ -runners, as introduced by Hanusa-Jones [10]. Under this bijection $F_{\mathcal{A}}$, the map Φ_n induces a map $\mathcal{A}_{2n} \rightarrow \mathcal{A}_{2n-2}$ on abacus diagrams with $(2n)$ runners to those with $(2n - 2)$ runners. In addition, there is a bijection $F_{\mathcal{R}}$ from abacus diagrams on $(2n)$ -runners to lattice points in \mathbb{Z}^n . We are able to explicitly describe the bijections Φ_n^k and their (co)domains in terms of all of these combinatorial models for the parabolic quotient \tilde{C}_n/C_n in a manner which makes the diagram below commute. It turns out that the induced map Φ_n on abacus diagrams is the most natural to describe. Theorem A is then proved by translating the condition imposed on symmetric cores to the corresponding abacus diagrams and coroot lattice points.

$$\begin{array}{ccccc}
\mathcal{S}_{2n} & \xrightarrow{F_{\mathcal{A}}} & \mathcal{A}_{2n} & \xrightarrow{F_{\mathcal{R}}} & \mathcal{R}_{2n} \\
\downarrow \Phi_n & & \downarrow \Phi_n & & \downarrow \Phi_n \\
\mathcal{S}_{2n-2} & \xrightarrow{F_{\mathcal{A}}} & \mathcal{A}_{2n-2} & \xrightarrow{F_{\mathcal{R}}} & \mathcal{R}_{2n-2}
\end{array}$$

We also interpret the bijections Φ_n^k both algebraically and geometrically. In terms of reduced words in the quotient of the corresponding Coxeter group \tilde{C}_n/C_n , the result we obtain is the following.

Theorem B (Corollaries 6.12 and 7.3). *For any reduced word in \tilde{C}_n/C_n corresponding to a symmetric $(2n)$ -core with first part equal to k , applying Φ_n decreases the length of the word by exactly k .*

The novelty of Theorem B lies more in the method of its proof, which uses delicate geometric arguments on the associated affine hyperplane arrangement. We also remark that Theorem B is one of many results obtained in this paper which is not a direct generalization of the work in [1]. In fact, we provide two distinct proofs of Theorem B, one geometric and the other algebraic. The geometric proof uses the theory of alcoves and coroot lattice points to study Φ_n . The correspondence between the coroot lattice and reduced words gives an induced action of Φ_n on alcoves in \mathbb{R}^n . Under this correspondence, the lattice points of the alcoves corresponding to elements of \mathcal{S}_{2n}^k all lie on a single hyperplane. Moreover, when we identify this hyperplane with the Euclidean space \mathbb{R}^{n-1} associated to \tilde{C}_{n-1} , the map Φ_n^k may be realized as a geometric projection of the alcoves onto the hyperplane containing their coroot lattice points. In particular, using the correspondence between reduced



A visualization of Φ_2^{12} as a projection from \tilde{C}_2/C_2 to \tilde{C}_1/C_1 .

words and minimal length alcove walks, Theorem B is a consequence of the following.

Theorem C (Theorem 6.11). *Let w be a minimal length coset representative for \tilde{C}_n/C_n such that the symmetric core partition corresponding to w has first part equal to k . If*

$$\mathcal{A}^1 \rightarrow \cdots \rightarrow \mathcal{A}^r$$

is an alcove walk for w , then

$$(1.1) \quad \pi(\mathcal{A}^1) \rightarrow \cdots \rightarrow \pi(\mathcal{A}^r)$$

is an alcove walk for $\Phi_n(w)$. Here, π is the projection onto the hyperplane containing the coroot lattice points of the symmetric $(2n)$ -core partitions with first part k . Moreover, if one removes all repetitions of the alcoves in (1.1), the resulting walk is a minimal length alcove walk for $\Phi_n(w)$.

On the other hand, the algebraic proof of Theorem B presents an explicit description of Φ_n on reduced words. Given a minimal length coset representative $w \in \tilde{C}_n/C_n$, there is an action of the reduced word on the abacus it defines. We analyze this action on the corresponding abacus and provide an explicit algorithm that constructs a reduced word for $\Phi_n(w)$ in $\ell(w)$ -steps.

The map Φ_n also exhibits other nice combinatorial properties which suggest applications to other areas of mathematics, particularly in the direction of affine Schubert calculus. For example, we have the following result on how Φ_n preserves strong Bruhat order.

Theorem D (Theorem 8.2). *Given two elements x and y in \tilde{C}_n/C_n whose associated coroot lattice points lie on the same hyperplane domain of Φ_n^k , then $x \geq_B y$ if and only if $\pi(x) \geq_B \pi(y)$.*

To prove Theorem D, we use the equivalence between the containment of core partitions and domination in the strong Bruhat order, which is introduced in section 5.3 of Hanusa-Jones [10] in answer to a question of Billey and Mitchell [3].

1.2. Applications and directions for future work. The authors strongly suspect that analogous results can be achieved in Lie types B and D , although no such generalizations have yet been formulated. One primary difference in these types is that the core partitions of Hanusa-Jones have dynamic residue labelings, which makes the bijection on core partitions difficult to conjecture. Therefore, in types B and D the geometry of the alcove model will once more be absolutely crucial not only for proving results, but also for even formulating statements. Another additional difficulty with a direct generalization from type C is that the domains for the projections provably do not lie on any of the root hyperplanes themselves. This was particularly surprising for the authors to discover about type B , since combinatorially type B is a subset of type C , and geometrically they are dual. However, much of the groundwork for a geometric analysis of types B and D has been laid in Section 6.

There are also directions for future work on the combinatorics of many related partially ordered sets. As Theorem D suggests, a great deal of combinatorial structure is preserved by the bijective projection developed in this paper. In addition to strong Bruhat order, there are other natural partial orderings to consider on these parabolic quotients, and it would be interesting to more explicitly describe which intervals comprise the domains and images of these bijections, for example.

Although the statements of many of the results of this paper are combinatorial in nature, both the motivation for the work and the most promising future directions are either algebro-geometric or representation-theoretic. The homology of the affine Grassmannian has a basis of Schubert classes which are indexed by elements of the parabolic quotients studied in this paper. The fact that the projection map preserves strong Bruhat order means that Schubert cells in one dimension map to Schubert cells one dimension lower. Therefore, the bijective projections developed in [1] and this paper may yield a means by which one can construct inductive proofs in affine Schubert calculus. In [21], Ram discusses how the root operators \tilde{e}_i and \tilde{f}_i coincide with the rank one crystal base operators after projection onto the line orthogonal to a hyperplane determined by the simple root α_i . It would be interesting to see if the projections defined in this paper can be similarly interpreted into the language of affine crystals.

The geometric results on alcove walks in Section 6 in particular may have other potential applications in algebraic geometry and representation theory. In the study of Shimura varieties with Iwahori level structure, Hanies and Ngô develop the notion of an alcove walk in the w -direction [9]. The alcove walk algebra, developed by Ram in [21] as a refinement of the Littelmann path model of [18], provides a combinatorial method for working with the affine Hecke algebra. This model

had already been used by Schwer to provide a combinatorial description of the Hall-Littlewood polynomials [23], and Ram and Yip further apply the alcove walk algebra to the theory of Macdonald polynomials in [22]. Parkinson, Ram, and Schwer present a refinement of Ram's alcove walk model in order to study analogs of Mirković-Vilonen cycles in the affine flag variety [20]. Essential to the work on alcove walks in these various contexts is the ability to additionally track information about the direction or orientation of various parts of the walk. Potential applications of the work in this paper to Shimura varieties, Mirković-Vilonen cycles, Macdonald polynomials, and the affine Hecke algebra therefore arise from the refined information obtained in Section 6 on so-called perpendicular, parallel, and diagonal steps in alcove walks.

1.3. Organization of the paper. This paper is written so that it is completely self-contained, and so most of the content in the first several sections is a summary of standard material. In Section 2 we provide a review of the language of root systems, Weyl groups and their parabolic quotients, and the affine hyperplane arrangement which gives rise to the alcove model for affine Weyl groups. In Section 3, we review the combinatorial models for the minimal length coset representatives of the parabolic quotient \widetilde{W}/W , including a self-contained overview of the relevant combinatorics in other classical Lie types from Hanusa-Jones [10]. We summarize in Section 4 the results obtained in [1] by Berg, Jones, and Vazirani for the case of $W = S_n$.

The new results begin in Section 5, where we develop the map Φ_n on the type C quotient and prove Theorem A. Section 6 develops the geometry of Φ_n using the alcove model. In particular, we extend the result obtained in Section 5 which says that the domains of Φ_n may be partitioned into hyperplanes, and show that when we identify these hyperplanes with \mathbb{R}^{n-1} , then Φ_n may be regarded as a projection from alcoves of \widetilde{C}_n/C_n in \mathbb{R}^n to alcoves of $\widetilde{C}_{n-1}/C_{n-1}$ in \mathbb{R}^{n-1} . This geometric interpretation provides a constructive proof for Theorem C. In Section 7, we provide an explicit algorithm that determines the action of Φ_n on reduced words of \widetilde{C}_n/C_n . The geometry developed in Section 6 and the algorithm provided in Section 7 give two distinct proofs of Theorem B. Finally, in Section 8, we prove Theorem D, showing that Φ_n preserves the strong Bruhat order on each of its hyperplane domains.

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2. WEYL GROUPS, ROOT SYSTEMS, AND ALCOVES

We begin by establishing some notation and providing a brief review of Coxeter groups, Weyl groups, and their associated root systems, largely following [4] and [11].

Definition 2.1. A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ for $s \neq s' \in S$. If no relation occurs for a pair s, s' , then set $m(s, s') = \infty$. Given this presentation, we may refer to W itself as a *Coxeter group*.

Since the generators $s \in S$ have order 2 in W , each $w \neq 1$ in W can be written as $w = s_1 s_2 \cdots s_r$ for some s_i (not necessarily distinct) in S . When r is as small as possible, we call it the length of w , written as $\ell(w)$. The expression of w as a product of $\ell(w)$ elements of S is called a *reduced expression*, or a *reduced word*. Some elementary properties of the length function are found in Section 1.4 of [4].

We will now introduce the Bruhat order, a partial order of W that is compatible with the length function.

Definition 2.2. Let (W, S) be a Coxeter system and $T = \{wsw^{-1} : w \in W, s \in S\}$ its set of reflections. Write

- (i) $w' \rightarrow w$ if $w'^{-1}w = t \in T$ and $\ell(w') < \ell(w)$.
- (ii) $w' < w$ if there is a sequence such that $w' = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_{k-1} \rightarrow w_k = w$.

The *Bruhat order* is the partial order relation on the set W defined by (ii).

Some basic combinatorial properties of the Bruhat order are found in Chapter 2 of [4].

Definition 2.3. Let (W, S) be a Coxeter system. If $m(s, s') = 2, 3, 4, 6$ for all $s \neq s' \in S$, then the corresponding Coxeter group W is called a *Weyl group*.

Weyl groups are examples of finite reflection groups. Abstractly, when they are not considered as subgroups of a linear group, they are finite Coxeter groups. Hence, they can be classified by their Coxeter-Dynkin diagrams. The Weyl groups of interest in this paper are A_n , B_n , C_n , and D_n . Their Dynkin diagrams are shown in Figure 1 (note that the Dynkin Diagrams for B_n and C_n are the same, for their root systems are conjugates of each other).

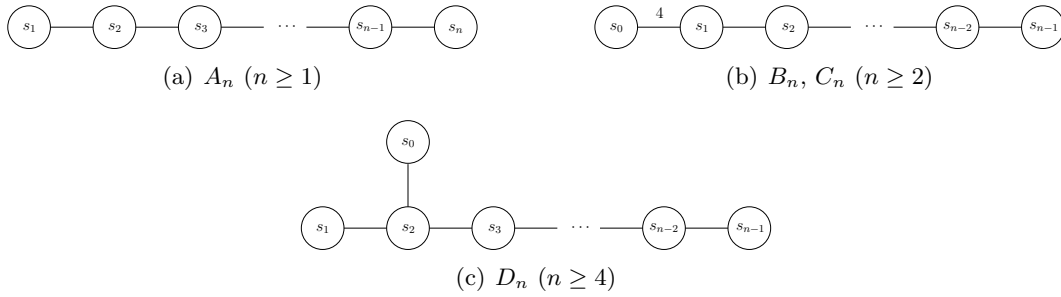


FIGURE 1. Dynkin diagrams for the Weyl groups A_n , B_n , C_n , and D_n .

We may now view W as a finite reflection group, acting on the Euclidean space V . In order to understand the internal structure of W , we introduce the notion of root systems. For any $\alpha \in V$, let s_α denote the reflection across the hyperplane perpendicular to α that passes through the origin. Following the definition given in Section 2.9 of [11], we have the following definition.

Definition 2.4. Let Φ be a finite set of nonzero vectors in V satisfying the following conditions:

- (i) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.
- (ii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.
- (iii) $\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Then Φ is called a (*crystallographic*) *root system*, and W , the group generated by all reflections s_α , $\alpha \in \Phi$, is called the *Weyl group* of Φ .

In Section 1.2 of [11], it is noted that any finite reflection group can be realized this way (possibly for many different choices of Φ), and, conversely, any group W arising from such root system is in fact finite.

Definition 2.5. A subset Δ of Φ is called a *simple system* if the following conditions are satisfied:

- (i) Δ is a vector space basis for the \mathbb{R} -span of Φ in V .
- (ii) Each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign.

The elements of a simple system Δ are called *simple roots*.

It is a fact that in a crystallographic root system, all roots are \mathbb{Z} -linear combinations of Δ , and the \mathbb{Z} -span of Δ in V is a W -stable lattice. The constructions of root systems of types A_n , B_n , C_n and D_n are found in Section 2.10 of [11]. We now introduce some terminologies regarding root systems that we will need later (for more facts and terminologies, cf. [5] and Section 2.9 of [11]):

- (i) Setting $\alpha^\vee := 2\alpha/\langle\alpha, \alpha\rangle$, the set Φ^\vee of all *coroots* α^\vee , $\alpha \in \Phi$, is also a crystallographic root system in V . It has the simple system $\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$. It is called the *dual* root system. The Weyl group of Φ^\vee is W , with $w\alpha^\vee = w(\alpha)^\vee$. As an example, the root systems of types B_n and C_n are dual to each other, hence giving isomorphic Weyl groups.
- (ii) The \mathbb{Z} -span Λ_R of Φ in V is called the *root lattice*. It is a lattice in the subspace of V spanned by Φ . Similarly, we may define the *coroot lattice* Λ_R^\vee . Both lattices are W -stable.
- (iii) The *weight lattice* is defined to be

$$\hat{L}(\Phi) := \{\lambda \in V \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$

and the *coweight lattice* is defined to be

$$\hat{L}(\Phi^\vee) := \{\lambda \in V \mid \langle\lambda, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

The weight lattice $\hat{L}(\Phi)$ contains Λ_R as a subgroup of finite index f , and, similarly, $\hat{L}(\Phi^\vee)$ contains Λ_R^\vee as a subgroup of finite index f . Here, f is the determinant of the matrix of Cartan integers $\langle\alpha, \beta^\vee\rangle$ ($\alpha, \beta \in \Delta$).

We are now ready to introduce affine reflection groups, a class of infinite groups generated by affine reflections in the Euclidean space V . Our treatment of this material closely follows Section 4.1 – 4.5 of [11].

To start, define the *affine group* $\text{Aff}(V)$ to be the group consisting of all affine reflections across hyperplanes in V . It is shown in Section 4.1 of [11] that $\text{Aff}(V)$ is the semidirect product of $\text{GL}(V)$ and the group of translations of elements of V . For each root $\alpha \in \Phi$ and each integer k , consider the affine hyperplane

$$H_{\alpha,k} := \{\lambda \in V \mid \langle\lambda, \alpha\rangle = k\}.$$

The corresponding affine reflection across this hyperplane is

$$s_{\alpha,k}(\lambda) := \lambda - (\langle\lambda, \alpha\rangle - k)\alpha^\vee.$$

Note that $H_{\alpha,k} = H_{-\alpha,-k}$ and $H_{\alpha,0}$ is the hyperplane H_α (so $s_{\alpha,0} = s_\alpha$).

Let \mathcal{H} be the collection of all hyperplanes $H_{\alpha,k}$, $\alpha \in \Phi$, $k \in \mathbb{Z}$. The following proposition, which may be verified by direct computation, shows that elements of \mathcal{H} are permuted naturally by elements of W and certain translations in $\text{Aff}(V)$.

Proposition 2.6.

- (i) If $w \in W$, then $wH_{\alpha,k} = H_{w\alpha,k}$ and $ws_{\alpha,k}w^{-1} = s_{w\alpha,k}$.
- (ii) If $\lambda \in V$ satisfies $\langle\lambda, \alpha\rangle \in \mathbb{Z}$ for all roots α , then $t(\lambda)H_{\alpha,k} = H_{\alpha,k+\langle\lambda, \alpha\rangle}$ and $t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k+\langle\lambda, \alpha\rangle}$.

Define the *affine Weyl group* \widetilde{W} to be the subgroup of $\text{Aff}(V)$ generated by all affine reflections $s_{\alpha,k}$, where $\alpha \in \Phi$, $k \in \mathbb{Z}$. The following proposition gives the structure of \widetilde{W} .

Proposition 2.7. \widetilde{W} is the semidirect product of W and the translation group corresponding to the coroot lattice $L = \Lambda_R^\vee$.

Proof. See proof of Proposition 4.2 in [11]. □

It is a fact that elements of \widetilde{W} permute the hyperplanes in \mathcal{H} . Thus, they permute the collection \mathcal{A} of connected components of $V^\circ := V \setminus \bigcup_{H \in \mathcal{H}} H$. Each element of \mathcal{A} is called an *alcove*. Suppose the root system Φ is irreducible. Fix a set Δ of simple roots in Φ . The *fundamental alcove* A_\circ is the alcove

$$A_\circ = \{\lambda \in V \mid 0 < (\lambda, \alpha) < 1 \text{ for all } \alpha \in \Phi^+\},$$

where Φ^+ is the set of positive roots. In fact, any alcove A consists of all $\lambda \in V$ satisfying the strict inequalities $k_\alpha < (\lambda, \alpha) < k_\alpha + 1$, where α runs through Φ^+ and $k_\alpha \in \mathbb{Z}$.

The *walls* of A_\circ are hyperplanes H_α , $\alpha \in \Delta$ and $H_{\tilde{\alpha},1}$, where $\tilde{\alpha}$ is the unique highest root $\tilde{\alpha}$ (such unique highest root exists because Φ is irreducible). Now, define \tilde{S} to be the set of reflections

$$\tilde{S} := \{s_\alpha, \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}.$$

The following proposition shows that \widetilde{W} acts transitively on \mathcal{A} .

Proposition 2.8. *The group \widetilde{W} permutes the collection \mathcal{A} of all alcoves transitively, and is generated by the set S_α of reflections with respect to the walls of the alcove A_\circ .*

Proof. See proof of Proposition 4.3 in [11]. □

Since \tilde{S} generates \widetilde{W} , it is natural to define the length $\ell(w)$ of an element $w \in \widetilde{W}$ to be the smallest r such that w is a product of r elements of \tilde{S} . Such expression is called *reduced*. For our Weyl groups, namely the Weyl groups A_n , B_n , C_n and D_n , the reflections $\{s_\alpha, \alpha \in \Delta\}$ correspond to their generators, and the reflection $s_{\tilde{\alpha},1}$ corresponds to the “extra generator” that we add in to make the affine Weyl groups \tilde{A}_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n .

Example 2.9. The Weyl group C_2 has the presentation

$$C_2 = \langle s_1, s_2 : (s_1 s_2)^4 = 1 \rangle,$$

and the affine Weyl group \tilde{C}_2 has the presentation

$$(2.1) \quad \tilde{C}_2 = \langle s_0, s_1, s_2 : (s_0 s_1)^4 = (s_0 s_2)^2 = (s_1 s_2)^4 = 1 \rangle.$$

In Figure 2, the walls of the fundamental alcove A_\circ are labeled with the generators of \tilde{C}_2 . The reflections generated by s_0, s_1, s_2 permute A_\circ transitively.

We will now give a geometric interpretation of length functions. Given a hyperplane $H = H_{\alpha,k} \in \mathcal{H}$, each alcove lies in one of the two half-spaces defined by H . We say that H *separates* two alcoves A and A' if these alcoves lie in different half-spaces relative to H . For example, the hyperplane H_s separates A_\circ and sA_\circ for all $s \in \tilde{S}$.

Given a pair of alcoves, it is clear that the number of hyperplanes that separate them is finite, for any such hyperplane must intersect any segment connecting the two alcoves, and a segment connecting the two alcoves only intersect a finite number of hyperplanes. For any $w \in \widetilde{W}$, let $n(w)$ be the cardinality of the set

$$\mathcal{L}(w) := \{H \in \mathcal{H} \mid H \text{ separates } A_\circ \text{ and } wA_\circ\}.$$

We have the following theorem.

Theorem 2.10. *For all $w \in \widetilde{W}$, $n(w) = \ell(w)$.*

Proof. See the discussion in Section 4.4 and 4.5 of [11]. □

An *alcove walk* from A_\circ to wA_\circ is a path connecting a point in the interior of A_\circ to a point in the interior of wA_\circ , with another condition that the path cannot pass through the vertex of any alcove. Consider all such paths from A_\circ to wA_\circ . Let r be the minimum number of hyperplanes in \mathcal{H} that such path intersects. It is an easy corollary of Theorem 2.10 that $\ell(w) = r$. We call

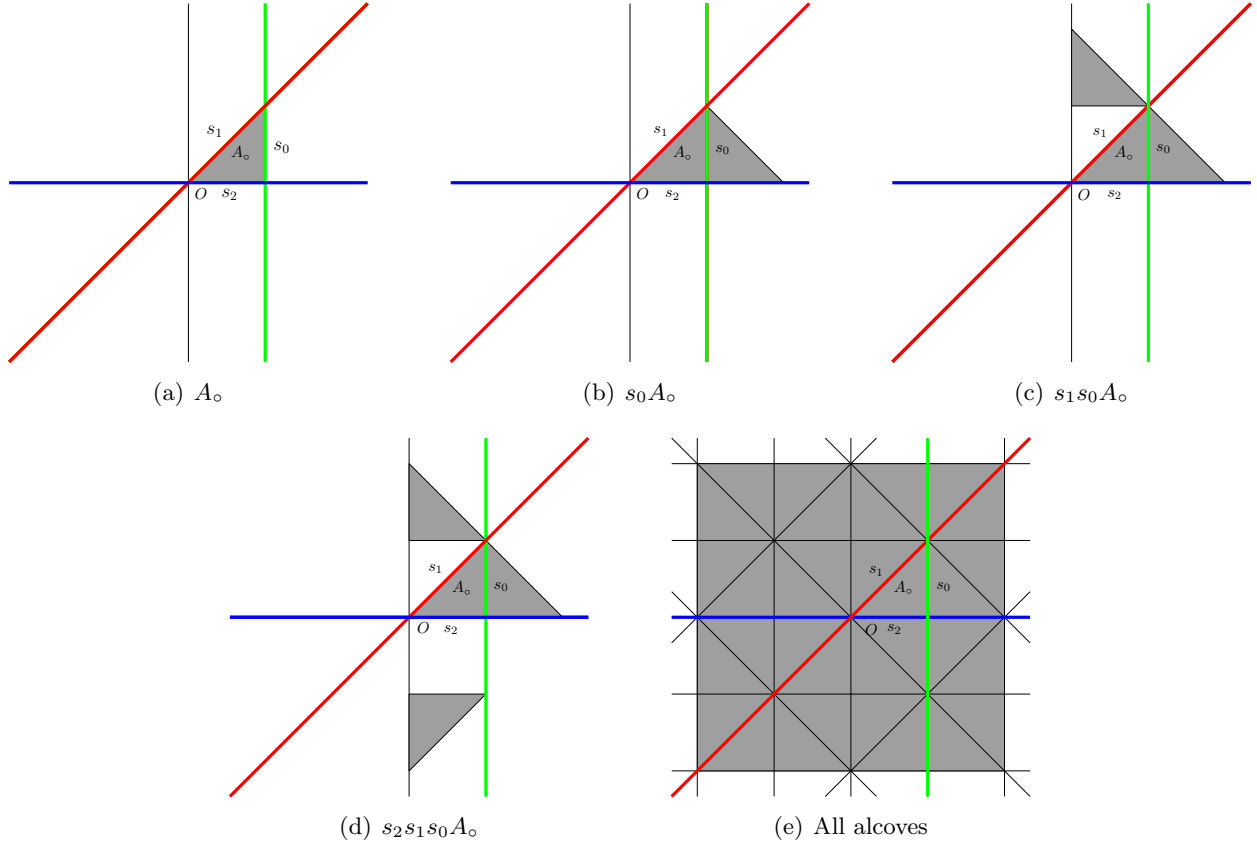


FIGURE 2. Elements of \tilde{C}_2 permute A_o transitively.

such a walk the *minimal length alcove walk* from A_o to wA_o (the minimal length alcove walk is not necessarily unique).

It is a general fact (cf. [21], for example) that we may label the hyperplanes in \mathcal{H} with generators in S_α , so that if an alcove walk from A_o to A' passes through the hyperplanes labeled $s_{i_1}, s_{i_2}, \dots, s_{i_k}$, in that order, then the alcove A' is the alcove wA_o , where $w = s_{i_1}s_{i_2} \cdots s_{i_k}$.

Example 2.11. Consider the affine Weyl group \tilde{C}_2 . Figure 3 shows two alcove walks beginning at A_o and ending at the same alcove. The first alcove walk is a minimal alcove walk, and the word corresponding to it is $w = s_2s_1s_2s_0s_1s_0$. The second alcove walk is not a minimal alcove walk. The word corresponding to the second alcove walk is $w' = s_1s_2s_0s_1s_0s_1s_0s_1s_2s_1s_0s_1$. The two words w and w' are equal by using the defining relations of \tilde{C}_2 (2.1):

$$\begin{aligned}
s_1s_2s_0s_1s_0s_1s_0s_1s_2s_1s_0s_1 &= s_1s_2s_1s_0s_2s_1s_0s_1 \\
&= s_1s_2s_1s_2s_0s_1s_0s_1 \\
&= s_2s_1s_2s_1s_0s_1s_0s_1 \\
&= s_2s_1s_2s_1s_0s_1s_0 \\
&= s_2s_1s_2s_0s_1s_0
\end{aligned}$$

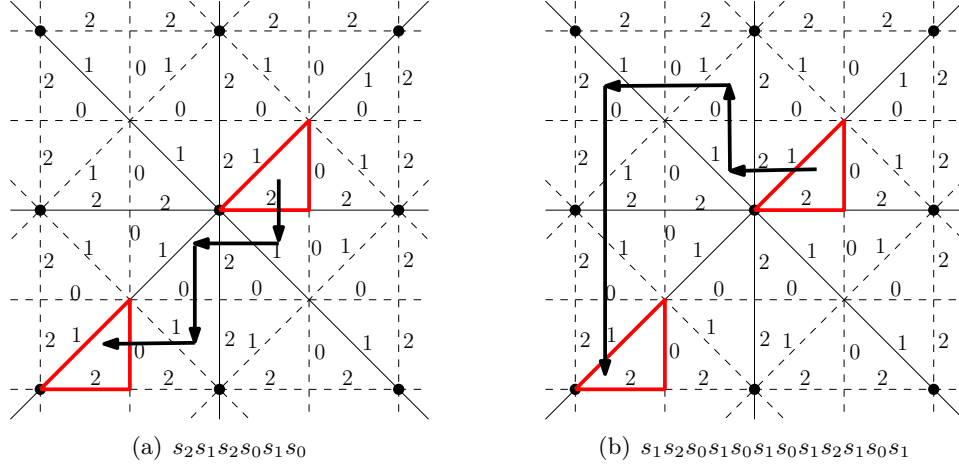


FIGURE 3. Examples of alcove walks.

3. COMBINATORIAL MODELS FOR THE PARABOLIC QUOTIENT \widetilde{W}/W

Having established the geometric interpretation of the length function on the affine Weyl group \widetilde{W} , we move on to consider the affine quotient \widetilde{W}/W , the object of our interest in this paper. Our goal is to find projection maps $\widetilde{W}_n \rightarrow \widetilde{W}_{n-1}$, where $W_n \in \{B_n, C_n, D_n\}$. To index the quotient, we consider the minimal length element in each coset. It is a well-known fact that each coset has a unique minimal length representative element.

We introduce three closely related combinatorial models that index minimal length coset representatives of the quotient. They are root lattice point model, abacus diagrams, and core partitions. In the following discussion of the models, we denote the affine Weyl group by \widetilde{W}_n , and the finite Weyl group by W_n .

3.1. Root Lattice Point Model. To start, recall that \widetilde{W}_n is the semi-direct product of W_n and the coroot lattice points Λ_R^\vee , i.e. $\widetilde{W}_n = \Lambda_R^\vee \rtimes W_n$. We may also identify the Euclidean space V with $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda_R^\vee$.

Let w be an element in \widetilde{W}_n . Define the coroot lattice point of w to be the result of acting on $0 \in V$ by w . Since elements in W_n leave $0 \in V$ unchanged, two elements in the same coset of \widetilde{W}_n/W_n send $0 \in V$ to the same coroot lattice point. Hence, there is a correspondence between coroot lattice points and cosets of \widetilde{W}_n/W_n .

Remark 3.1. It should be pointed out that the authors of [1] and [10] have used the term “root lattice point” where we instead say “coroot lattice point.” This difference in terminology arises because in type A , which was studied by the authors of [1], the root and coroot lattices coincide. In the type C root system, however, we must use the coroot lattice specifically, on which there is a well-defined action of \widetilde{W}_n . To preserve the terminology in the existing literature, the authors have decided to refer to this model as the “root lattice point model.”

Geometrically, if \mathfrak{a}^\vee is the coroot lattice point corresponding to a coset, then the union of the alcoves that represent elements in that coset is a translation of the fundamental region (the union of the alcoves representing the elements of W_n) by \mathfrak{a}^\vee . The minimal length coset representative is the alcove that is closest to the fundamental alcove (see previous section on alcove walks). Figure 4 shows the minimal length coset representatives for \widetilde{C}_2/C_2 . The bijections between cosets of \widetilde{W}_n/W_n

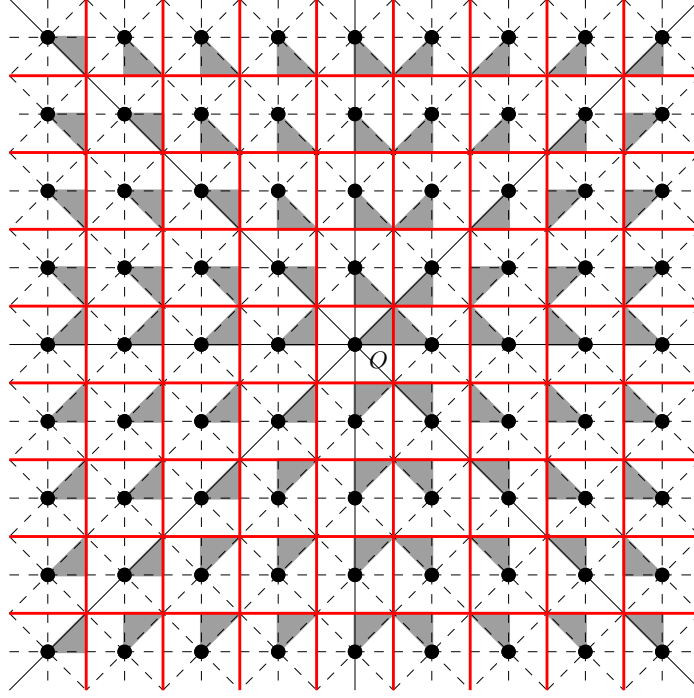


FIGURE 4. Minimal length coset representatives for \tilde{C}_2/C_2 .

and coroot lattice points for $W_n \in \{B_n, C_n, D_n\}$ are found in Section 4 of [10]. The bijections are summarized in the table below (cf. Table 3 of [10]):

Type	Coroot Lattice Points
\tilde{B}_n/B_n	$(a_1, \dots, a_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n a_i $ is even.
\tilde{C}_n/C_n	$(a_1, \dots, a_n) \in \mathbb{Z}^n$
\tilde{D}_n/D_n	$(a_1, \dots, a_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n a_i $ is even.

3.2. Abacus Diagrams. Our development of abacus diagrams is based on Section 2 and Section 3 of [10]. To define abacus diagrams, we first introduce mirrored \mathbb{Z} -permutations.

Definition 3.2. (Definition 2.1 in [10]) Fix a positive integer n and let $N = 2n + 1$. A bijection $w : \mathbb{Z} \rightarrow \mathbb{Z}$ is a *mirrored \mathbb{Z} -permutation* if $w(i + N) = w(i) + N$ and $w(-i) = -w(i)$ for all $i \in \mathbb{Z}$.

It is easy to see from the equations in Definition 3.2 that a mirrored \mathbb{Z} -permutation w is completely determined by its action on $\{1, 2, \dots, n\}$, and that $w(i) = i$ for all $i \equiv 0 \pmod{N}$. Furthermore, as discussed in Section 8 of [4], every element in \tilde{B}_n , \tilde{C}_n and \tilde{D}_n can be represented by such mirrored \mathbb{Z} -permutation. The mirrored \mathbb{Z} -permutations corresponding to the Coxeter generators of \tilde{B}_n , \tilde{C}_n , and \tilde{D}_n are shown below:

$$\begin{aligned}
s_i &= (1, 2, \dots, i-1, i+1, i, i+2, \dots, n) \text{ for } 1 \leq i \leq n-1, \\
s_0^C &= (-1, 2, 3, \dots, n), \\
s_n^C &= (1, 2, \dots, n-1, n+1), \\
s_0^D &= (-2, -1, 3, 4, \dots, n), \\
s_n^D &= (1, 2, \dots, n-2, n+1, n+2)
\end{aligned}$$

The conditions on mirrored \mathbb{Z} -permutations for elements of \tilde{B}_n , \tilde{C}_n , and \tilde{D}_n are found in Section 8 of [4]. They are summarized in the table below (cf. Theorem 2.3 in [10]):

Type	Mirrored \mathbb{Z} -permutation	Minimal length coset representative in \tilde{W}_n/W_n
\tilde{B}_n	$ i \in \mathbb{Z} : i \leq 0, w(i) \geq 1 \equiv 0 \pmod{2}$	$w(1) < w(2) < \cdots < w(n) < w(n+1)$
\tilde{C}_n	all mirrored \mathbb{Z} -permutations	$w(1) < w(2) < \cdots < w(n) < w(n+1)$
\tilde{D}_n	$ i \in \mathbb{Z} : i \leq 0, w(i) \geq 1 \equiv 0 \pmod{2}$ and $ i \in \mathbb{Z} : i \leq n, w(i) \geq n+1 \equiv 0 \pmod{2}$	$w(1) < w(2) < \cdots < w(n) < w(n+2)$

Given a mirrored \mathbb{Z} -permutation w , the ordered sequence $[w(1), w(2), \dots, w(2n)]$ is called the *base window* of w . The left action of \tilde{W}_n on the base windows interchanges values. The following lemma, called the balance lemma, is needed later in this section to establish the correspondence between mirrored \mathbb{Z} -permutations and abacus diagrams:

Lemma 3.3 (Balance Lemma). *If w is an mirrored \mathbb{Z} -permutation, then $w(i) + w(N - i) = N$ for all $i = 1, 2, \dots, n$. Conversely, if $w : \mathbb{Z} \rightarrow \mathbb{Z}$ is a bijection that satisfies $w(i + N) = w(i) + N$ and $w(i) + w(N - i) = N$, then w is a mirrored \mathbb{Z} -permutation.*

Proof. See proof of Lemma 2.4 in [10]. □

Lemma 2.5 and 2.6 in [10] characterize the base windows for \tilde{W}_n/W_n , where $W_n \in \{B_n, C_n, D_n\}$. From these characterizations, we immediately arrive at the following theorem:

Theorem 3.4 (Corollary 2.7 in [10]). *Suppose $w, w' \in \tilde{W}_n/W_n$. If $w \neq w'$, then*

$$\{w(1), w(2), \dots, w(2n)\} \neq \{w'(1), w'(2), \dots, w'(2n)\}$$

as unordered sets.

Having characterized mirrored \mathbb{Z} -permutations corresponding to the minimal length coset representatives of \tilde{W}_n/W_n , we are finally ready to introduce abacus diagrams. The abacus diagrams combinatorialize the integers occurring in the base window of a mirrored \mathbb{Z} -permutation.

Definition 3.5. (Definition 3.1 in [10]) An *abacus diagram* is a diagram containing $2n$ columns labeled $1, 2, \dots, 2n$, called *runners*. Runner i contains entries labeled by the integers $mN + i$ for each level m where $-\infty < m < \infty$.

An abacus diagram is drawn as follows:

- (i) Each runner is vertical, with $-\infty$ at the top and ∞ at the bottom. The runners increase from runner 1 in the leftmost position to runner $2n$ in the rightmost position.
- (ii) Entries in the abacus diagram may be circled. The circled entries are called *beads*, and the non-circled entries are called *gaps*. The entries are linearly ordered by the labels $mN + i$, where $m \in \mathbb{Z}$ is the level and $1 \leq i \leq 2n$ is the runner number. This linear ordering is called the *reading order*.
- (iii) A bead b is *active* if there exist gaps that occur prior to b in the reading order. Otherwise, the bead b is *inactive*. A runner is called *flush* if no bead on the runner is preceded in reading order by a gap in the same runner. An abacus is flush if every runner is flush.
- (iv) An abacus is *balanced* if there is at least one bead on every runner, and the sum of labels of the lowest bead on runners i and $N - i$ is N for all $i = 1, 2, \dots, 2n$ (equivalently, the sum of the highest levels that contain a bead for runners i and $N - i$ is 0).
- (v) An abacus is *even* if there is an even number of gaps preceding N in the reading order.

An example of the abacus diagram is shown in Figure 5.

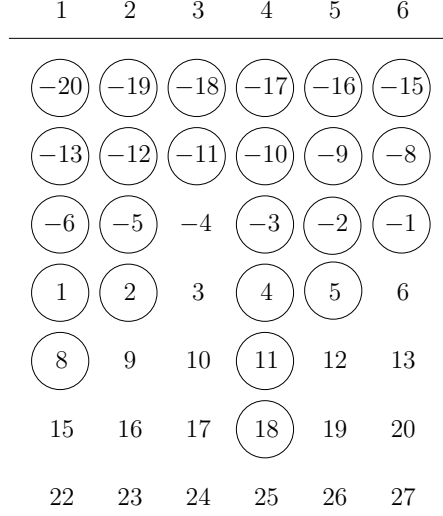


FIGURE 5. The balanced flush abacus $\mathbf{a}(w)$ corresponding to the mirrored \mathbb{Z} -permutations w whose base window is $w = [-11, -1, 2, 5, 8, 18]$.

Definition 3.6. Given a mirrored \mathbb{Z} -permutation w , define $\mathbf{a}(w)$ to be the flush abacus whose lowest bead in each runner is an element of $\{w(1), w(2), \dots, w(2n)\}$.

The abacus $\mathbf{a}(w)$ is well-defined because by Lemma 2.5 in [10], $w(1), \dots, w(2n)$ have distinct residues mod N , and none of them are equivalent to 0 mod N . Furthermore, by Lemma 3.3 (balance lemma), the abacus $\mathbf{a}(w)$ is a balanced abacus. From now on, we will assume all abacus diagrams to be balanced flush abacus diagrams unless otherwise noted.

The following lemma characterizes the set of abaci that corresponds to \widetilde{W}_n/W_n when $W_n \in \{B_n, C_n, D_n\}$.

Lemma 3.7. For $W_n \in \{B_n, C_n, D_n\}$, the map \mathfrak{A} is a bijection from \widetilde{W}_n/W_n to the set of abaci shown in the table below.

Type	Abaci
\widetilde{B}_n/B_n	even balanced flush abaci
\widetilde{C}_n/C_n	balanced flush abaci
\widetilde{D}_n/D_n	even balanced flush abaci

Proof. The lemma follows directly from the characterization of mirrored \mathbb{Z} -permutation for \widetilde{B}_n/B_n , \widetilde{C}_n/C_n , and \widetilde{D}_n/D_n . For details of the proof, see proof of Lemma 3.6 in [10]. \square

When we translate the action of the Coxeter generators on the mirrored \mathbb{Z} -permutations through the bijection \mathfrak{A} to abacus diagrams, we get an action of the Coxeter generators on abacus diagrams. They are described in Section 3.2 of [10]. Since we are going to use these actions later in our proofs, we summarize them again here:

- (i) s_i interchanges column i with column $i + 1$ and interchanges column $2n - i$ with column $2n - i + 1$, for $1 \leq i \leq n - 1$.
- (ii) s_0^C interchanges column 1 and $2n$, and then shifts the lowest bead on column 1 down one level towards ∞ , and shifts the lowest bead on column $2n$ up one level towards $-\infty$.

- (iii) s_0^D interchanges columns 1 and 2 with columns $2n - 1$ and $2n$, respectively, and then shifts the lowest beads on columns 1 and 2 down one level each towards ∞ , and shifts the lowest beads on columns $2n - 1$ and $2n$ up one level each towards $-\infty$.
- (iv) s_n^C interchanges column n with column $n + 1$.
- (v) s_n^D interchanges columns $n - 1$ and n with columns $n + 1$ and $n + 2$, respectively.

The following theorem shows that there is a bijection between coroot lattice points and abacus diagrams for \widetilde{W}_n/W_n .

Theorem 3.8 (Theorem 4.1 in [10]). *The coroot lattice point for an element $w \in \widetilde{W}/W$ is*

$$\sum_{i=1}^n \text{level}_i(\mathbf{a}(w)) \varepsilon_i,$$

where $\text{level}_i(\mathbf{a}(w))$ is the level of the lowest bead in column i of the abacus $\mathbf{a}(w)$.

Proof. When we identify the Coxeter generators that act on the abacus diagrams with the Coxeter generators that act on the coroot lattice points, it can be checked that the action of \widetilde{W}_n on coroot lattice points is the same as the action of \widetilde{W}_n on abacus diagrams. The theorem then follows because the bijection between abacus diagrams and coroot lattice points is an equivariant bijection. \square

3.3. Core Partitions. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n , and $\ell \geq 2$ be an integer. The Young diagram of λ is a collection of left justified boxes for which the number of boxes weakly decreases from λ_1 to λ_r as one moves down the rows. We will use (i, j) to denote the box located at the i th row and the j th column of the Young diagram of λ . The (i, j) th hook length of λ , denoted by $h_{(i,j)}^\lambda$, is the number of boxes to the right and below the (i, j) th box of λ , including the (i, j) th box itself. A Young diagram is *symmetric* if it is symmetric across the line formed by the boxes along the diagonal (i, i) . A partition λ is *even* if there is an even number of boxes on the main diagonal of λ .

Definition 3.9. A partition λ is a ℓ -core if for every box (i, j) in the Young diagram of λ , we have $\ell \nmid h_{(i,j)}^\lambda$.

Following the notation in [1], we denote the set of all ℓ -cores, \mathcal{C}_ℓ , the set of all ℓ -cores with first part k , \mathcal{C}_ℓ^k , and the set of all ℓ -cores with first part $\leq k$, $\mathcal{C}_\ell^{\leq k}$. Similarly, we will denote the set of all symmetric ℓ -cores, \mathcal{S}_ℓ , the set of all symmetric ℓ -cores with first part k , \mathcal{S}_ℓ^k , and the set of all symmetric ℓ -cores with first part $\leq k$, $\mathcal{S}_\ell^{\leq k}$. In some existing literature, ℓ -cores are also equivalently defined as partitions having no removable ℓ -rim hooks.

There is a bijection between balanced flush abacus diagrams with $2n$ runners and $(2n)$ -cores. The bijection $F_{\mathcal{S}} : \mathcal{A}_{2n} \rightarrow \mathcal{S}_{2n}$ is defined as follows:

Definition 3.10. (Definition 5.2 in [10]) Let $\mathbf{a} \in \mathcal{A}_{2n}$ be a balanced flush abacus with $2n$ runners and M active beads. Define $F_{\mathcal{S}}(\mathbf{a})$ to be the partition whose i -th row contains the same number of boxes as gaps that appear before the $(M - i + 1)$ th active bead in reading order.

It is shown in Section 5 of [10] that the image of $F_{\mathcal{S}}$ is indeed symmetric $(2n)$ -cores. With this bijection and the characterization of the abacus diagrams corresponding to \widetilde{W}_n/W_n , the core partitions corresponding to \widetilde{W}_n/W_n are summarized in the table below:

Type	Abacus diagrams	Core partitions
\widetilde{B}_n/B_n	even balanced flush abaci	even symmetric $(2n)$ -cores
\widetilde{C}_n/C_n	balanced flush abaci	symmetric $(2n)$ -cores
\widetilde{D}_n/D_n	even balanced flush abaci	even symmetric $(2n)$ -cores

Under the bijection $F_{\mathcal{J}}$ between abacus diagrams and cores, we get a natural action of \widetilde{W}_n on cores. We will only describe the action of \widetilde{C}_n on symmetric $(2n)$ -cores. The actions of \widetilde{B}_n and \widetilde{D}_n on even symmetric $(2n)$ -cores are more complicated. The interested readers may refer to Section 5 of [10] for more details.

To describe the elements of \widetilde{C}_n on symmetric $(2n)$ -cores, we begin by orienting \mathbb{N}^2 so that (i, j) corresponds to row i and column j in the Young diagram. Define the *residue* of a position in \mathbb{N}^2 to be

$$\text{res}(i, j) = \begin{cases} (j - i) \parallel (2n) & \text{if } 0 \leq (j - i) \parallel (2n) \leq n \\ 2n - ((j - i) \parallel (2n)) & \text{if } n < (j - i) \parallel (2n) < 2n \end{cases}$$

where $p \parallel q$ is the integer in $\{0, 1, \dots, q-1\}$ that is congruent to $p \bmod q$. For example, the residues for \widetilde{C}_2 are shown in Figure 6.

0	1	2	1	0	1	2
1	0	1	2	1	0	1
2	1	0	1	2	1	0
1	2	1	0	1	2	1
0	1	2	1	0	1	2
1	0	1	2	1	0	1
2	1	0	1	2	1	0

FIGURE 6. Residues for \widetilde{C}_2

We say that a box is *addable* to a partition λ if adding the box to the Young diagram of λ results in a partition. Similarly, a box is *removable* if removing the box from λ results in a partition. The following theorem, which follows from Theorem 5.8 in [10] when applied to \widetilde{C}_n/C_n , gives the action of \widetilde{C}_n on symmetric $(2n)$ -core partitions in terms of residues:

Theorem 3.11. *Let $s_i \in \widetilde{C}_n$ be a Coxeter generator of \widetilde{C}_n . If there exist addable i -boxes or removable i -boxes, then s_i acts on λ by adding all addable i -boxes to λ , or deleting all removable i -boxes from λ . If there are no addable or removable i -boxes in λ , then s_i does not change λ .*

Proof. See proof of Theorem 5.8 in [10]. □

Remark 3.12. It is a consequence of Theorem 3.11 that the resulting partition after applying s_i to λ is still a symmetric $(2n)$ -core. Also, to see that the process is well defined, notice that the partition λ is a $(2n)$ -core, so there cannot be both an addable i -box and a removable i -box.

Using Theorem 3.11, we may repeatedly delete removable boxes from a $(2n)$ -core λ to obtain the reduced word corresponding to λ .

Example 3.13. Taking the 4-core shown in Figure 6 and repeatedly applying Theorem 3.11 as shown in Figure 7, we obtain the reduced word $s_0 s_1 s_0 s_2 s_1 s_0$ corresponding to the core.

4. REVIEW OF THE RESULTS FOR TYPE A

In this section, we review results obtained for the map $\Phi_n : \widetilde{S}_n/S_n \rightarrow \widetilde{S}_{n-1}/S_{n-1}$. Combinatorially, core partitions, abacus diagrams, and root lattice points index elements of \widetilde{S}_n/S_n . These

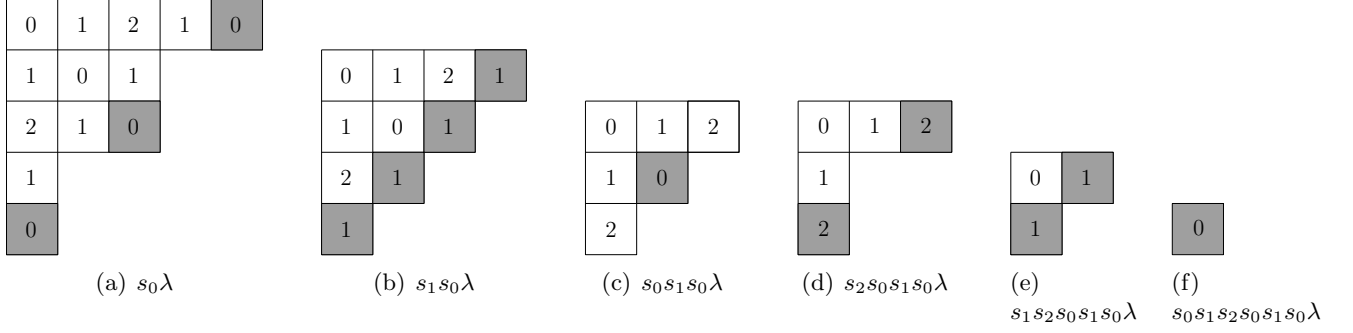


FIGURE 7. Action of \tilde{C}_2 on a 4-core.

models are the symmetric group analogues of the models described in Section 3 for types B_n , C_n , and D_n . Their conditions are summarized in the table below:

Type	Conditions
Core Partitions	n -cores
Abacus Diagrams	n runners. Sum of the highest levels that contain a bead equals 0.
Root Lattice Points	$(a_1, \dots, a_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n a_i = 0$.

We first define the map $\Phi_n^k : \tilde{S}_n/S_n \rightarrow \tilde{S}_{n-1}/S_{n-1}$ on core partitions. Given a n -core λ , consider its Young diagram. To apply Φ_n^k , we do the following: first, compute all $h_{(i,1)}$, the hook length of the left most square of the i th row. Then, delete all rows i of λ for which $h_{(i,1)} \equiv h_{(1,1)} \pmod{n}$. Using abacus diagrams, we can show that when Φ_n^k is applied to a n -core λ with first part equal to k , the resulting partition $\Phi_n^k(\lambda)$ is a $(n-1)$ -core with first part at most k . Furthermore, it is shown in [1] that the map $\Phi_n^k : \mathcal{C}_n^k \rightarrow \mathcal{C}_{n-1}^{\leq k}$ is a bijection.

The map Φ_n^k , when defined on root lattice points, becomes more geometrically enlightening. As a review, recall that the simple roots Δ of type A_{n-1} is the collection of $n-1$ vectors

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$

The n -cores correspond to root lattice points $(a_1, \dots, a_n) \in \Lambda_R^\vee$, where $a_i \in \mathbb{Z}$ and $\sum_{i=1}^n a_i = 0$. Let $V = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda_R^\vee \subsetneq \mathbb{R}^n$. Elements of V are $(a_1, \dots, a_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n a_i = 0$. The following result from [1] shows that when the cores are identified with the root lattice points, the domain \mathcal{C}_n^k of Φ_n^k lies inside a hyperplane in V .

Theorem 4.1 (Corollary 3.2.15 in [1]). *For $k \geq 0$, let H_ℓ^k denote the affine hyperplane*

$$H_\ell^k = \left\{ v \in \mathbb{R}^n : (v, \varepsilon_{(k \bmod n)}) = \left\lceil \frac{k}{n} \right\rceil \right\} \cap V$$

inside V , where $1 \leq (k \bmod n) \leq n$. Then under the correspondence between n -cores and root lattice points, the n -cores λ with $\lambda_1 = k$ all lie inside $H_n^k \cap \Lambda_R^\vee$.

Let π denote the correspondence between cores and root lattice points. Let ψ_n be the affine map defined by $\psi_n(a_1, \dots, a_n) = (a_n + 1, a_1, a_2, \dots, a_{n-1})$. It has been shown in [1] that the n -cores corresponding to the root lattice points (a_1, \dots, a_n) and $\psi_n(a_1, \dots, a_n)$ are the same. The next theorem shows that the map Φ_n^k , when restricted to the domain \mathcal{C}_n^k inside $H_n^k \cap \Lambda_R^\vee$, is a projection onto the hyperplane H_n^k .

Theorem 4.2 (Theorem 4.1.1 in [1]). *Let ψ_ℓ be the affine map as defined before. We have*

$$\pi^{-1} \circ \Phi_n^k \circ \pi(a_1, \dots, a_n) = \psi_{n-1}^{a_i}(a_1, \dots, \widehat{a_i}, \dots, a_n),$$

where a_n is the rightmost occurrence of the largest entry among (a_1, \dots, a_n) and the circumflex indicates omission.

Finally, it is shown that the bijection Φ_n^k has a natural description when expressed as a map between bounded partitions, under the correspondence

$$\rho_{n+1} : \{(n+1)\text{-cores}\} \rightarrow \{\text{partitions with first part } \leq n\}$$

of Lapointe and Morse [15]. In particular, under this correspondence, we obtain a map γ_n^k from partitions ν with $\nu_1 \leq n-1$ and $\text{len}(\nu) = k$ to partitions σ with $\sigma_1 \leq n-2$ and $\text{len}(\sigma) \leq k$. The map γ_n^k acts on ν by deleting the first column from the diagram of the partition. With γ_n^k , the following diagram is commutative:

$$\begin{array}{ccccc} \{\lambda \in \mathcal{C}_n^k\} & \xrightarrow{tr} & \{\lambda \in \mathcal{C}_n, \text{len}(\lambda) = k\} & \xrightarrow{\rho_n} & \{\nu : \nu_1 \leq n-1, \text{len}(\nu) = k\} \\ \downarrow \Phi_n^k & & \downarrow \tilde{\Phi}_n^k & & \downarrow \gamma_n^k \\ \{\mu \in \mathcal{C}_{n-1}^{\leq k}\} & \xrightarrow{tr} & \{\mu \in \mathcal{C}_{n-1}, \text{len}(\mu) \leq k\} & \xrightarrow{\rho_{n-1}} & \{\sigma : \sigma_1 \leq n-2, \text{len}(\sigma) \leq k\} \end{array}$$

In [15], the cores are connected to the expansions of k -Schur functions, which represent the Schubert basis in the homology of the affine Grassmanian. For more details, see Section 5 of [1] and Section 3 of [15].

In order to define the map $\widetilde{W}_n/W_n \rightarrow \widetilde{W}_{n-1}/W_{n-1}$ for other Lie types, it is necessary to use the theory of alcoves. We have seen in this section that for type A , the map Φ_n^k is a projection when restricted to the root lattice points corresponding to \mathcal{C}_n^k . In general, given \widetilde{W}_n/W_n , we wish to partition its elements into hyperplane domains. The cores corresponding to each hyperplane domain should satisfy common combinatorial properties. By identifying each hyperplane with $\widetilde{W}_{n-1}/W_{n-1}$, the map $\Phi_n : \widetilde{W}_n/W_n \rightarrow \widetilde{W}_{n-1}/W_{n-1}$ is subsequently defined by projecting each domain onto their hyperplanes. We will then proceed to analyze this map geometrically and combinatorially, using alcoves, core partitions, and abacus diagrams.

5. DEFINING THE MAP Φ FOR TYPE C

5.1. Defining the Map Φ on Symmetric Cores. The map Φ_n acting on the set \mathcal{S}_{2n} of symmetric $(2n)$ -cores can be defined in the following way. First, label the boxes (i, j) of a $(2n)$ -core by the residues $j - i \pmod{2n}$. Then, delete all the rows that end with the same residue as the first row. Finally, delete all the columns that end with the same residue as the first column. An example of this process is shown below:

5.2. The Map Φ on Abacus Diagrams. Given an abacus $\mathbf{a} = (a_1, \dots, a_n, -a_n, \dots, -a_1)$, define $\Phi'_n(\mathbf{a})$ to be the abacus obtained via the following procedure: first, in abacus \mathbf{a} , locate the rightmost runner with the largest coordinate. Then, delete this runner *and* its symmetric runner. The resulting abacus $\Phi'_n(\mathbf{a})$ is a balanced abacus with $2n - 2$ runners, corresponding to an element of $\widetilde{C}_{n-1}/C_{n-1}$. An example of this procedure is shown below (the abacus corresponds to the core partition in Figure 8):

It is shown in Hanusa-Jones [10] that balanced abacus diagrams for the quotient \widetilde{C}_n/C_n are in bijective correspondence with symmetric $(2n)$ -cores. Let this bijection be denoted by $F_{\mathcal{A}}$. Using this bijection and the map $\Phi_n : \mathcal{S}_{2n} \rightarrow \mathcal{S}_{2n-2}$ that we defined for symmetric core partitions, we

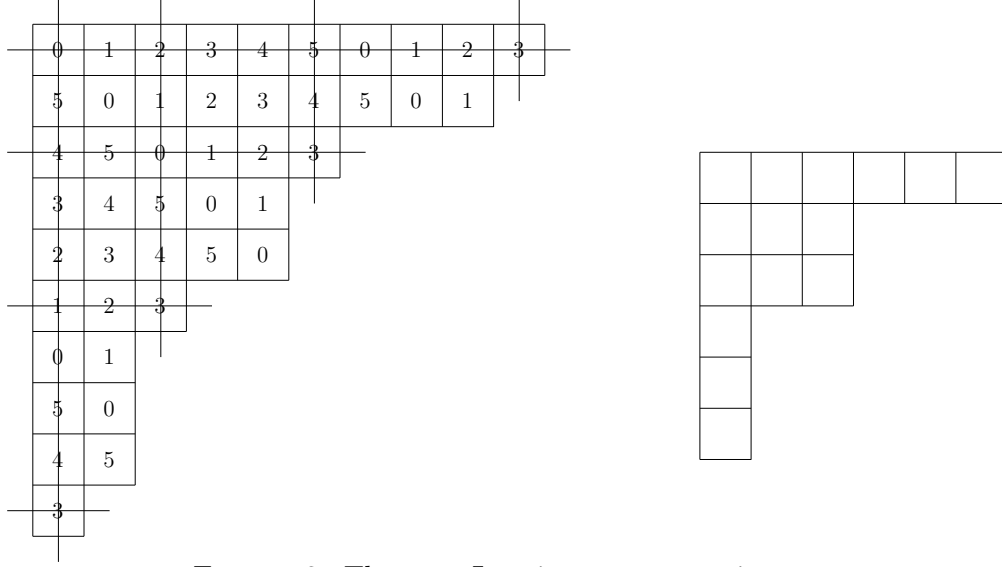


FIGURE 8. The map Φ acting on symmetric cores

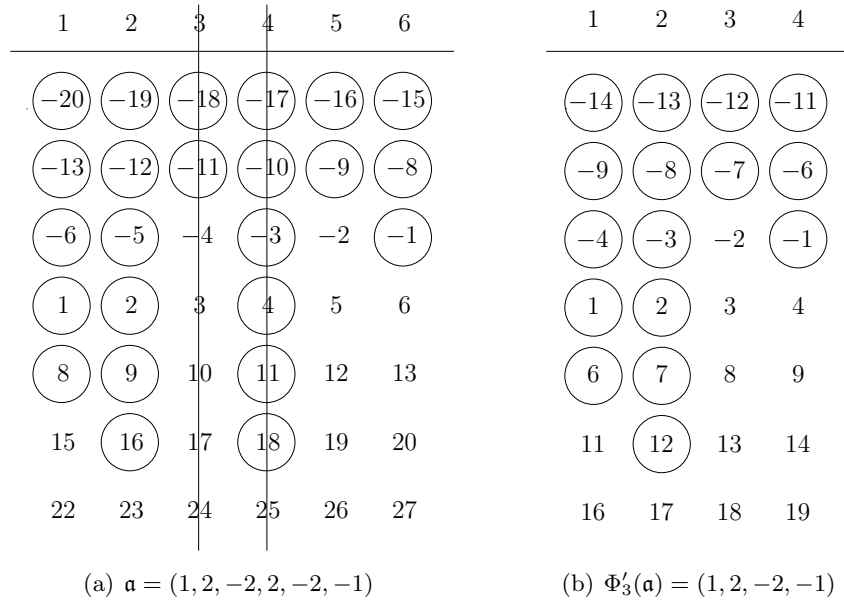


FIGURE 9. Action of Φ'_3 on the abacus $\mathbf{a} = (1, 2, -2, 2, -2, -1)$.

can induce the map $\Phi_n : \mathcal{A}_{2n} \rightarrow \mathcal{A}_{2n-2}$ on abacus diagrams such that the following diagram is commutative:

$$(5.1) \quad \begin{array}{ccc} \mathcal{S}_{2n} & \xrightarrow{F_{\mathcal{A}}} & \mathcal{A}_{2n} \\ \downarrow \Phi_n & & \downarrow \Phi_n \\ \mathcal{S}_{(2n-2)} & \xrightarrow{F_{\mathcal{A}}} & \mathcal{A}_{(2n-2)} \end{array}$$

Proposition 5.1. *The induced map Φ_n on abacus diagrams is the map Φ'_n .*

To prove this proposition, we will need the following lemma:

Lemma 5.2. *In an abacus diagram \mathbf{a} corresponding to the core partition $F_{\mathcal{J}}(\mathbf{a})$, there is a left-most runner whose largest bead is located at the lowest level. Deleting this runner results in an abacus \mathbf{a}' whose corresponding core partition is obtained from $F_{\mathcal{J}}(\mathbf{a})$ by deleting all columns that end with the same residue class as the first column.*

Proof. By construction of the core partition from the abacus diagram, the columns correspond to the gaps that are smaller than the largest active bead. For example, the smallest gap, which is smaller than every active bead, corresponds to the first column. It can be easily checked (similar to the case of deleting the right-most largest runner) that two columns end with the same residue class if and only if their gaps are in the same runner. The left-most runner whose largest bead is the lowest is the runner containing the smallest gap. By deleting this runner, we are deleting all the columns that end with the same residue class as the first column, as claimed. \square

We are now ready to prove Proposition 5.1.

Proof. Notice that the symmetric runner of the right-most largest runner is the left-most smallest runner. To apply Φ'_n , we will proceed in two steps. First, delete the right-most largest runner. This corresponds to deleting all rows that end in the same residue class as the first row. The resulting core-partition is a $(2n-1)$ -core. Now, delete the symmetric runner. By Lemma 5.2, this corresponds to deleting all columns in the $(2n-1)$ -core that end with the same residue class as the first column. As in the proof of Lemma 5.2, the columns correspond to gaps in the symmetric runner that are smaller than the largest active bead. Given a column that ends in the same residue class as the first column in the $(2n)$ -core, if the column is not deleted after the first step, then it still ends with the same residue class as the first column in the $(2n-1)$ -core. It follows that the resulting $(2n-2)$ -core partition we get from applying Φ'_n is what we would obtain if we apply the map Φ_n . This completes the proof. \square

5.3. The Domain and Codomain of the map Φ_n . In this section, we wish to find a partition of the domain of Φ_n so that when restricted to these parts, the map Φ_n is bijective.

Lemma 5.3. *Let $\mathbf{a} \in \mathcal{A}_{2n}$ be a balanced abacus with $2n$ runners. If the largest bead of \mathbf{a} is located at level ℓ of runner i , where $1 \leq i \leq 2n$, then the first part of the symmetric $(2n)$ -core $F_{\mathcal{J}}(\mathbf{a})$ is $\lambda_1 = 2n(\ell-1) + i$.*

Proof. The first part, λ_1 , of the $(2n)$ -core $F_{\mathcal{J}}(\mathbf{a})$ is equal to the number of gaps that are smaller than the largest bead. Let the largest bead of each runner be located at levels $(r_1, r_2, \dots, r_{2n}) = (a_1, \dots, a_n, -a_n, \dots, -a_1)$, respectively. Since the largest bead is located at level ℓ of runner i (i.e. $r_i = \ell$), the number of gaps that occur before this bead is

$$\begin{aligned} \sum_{j=1}^i (\ell - r_j) + \sum_{j=i+1}^{2n} (\ell - r_j - 1) &= 2n\ell - \sum_{j=1}^{2n} r_j - (2n - i) \\ &= 2n\ell - (2n - i) = 2n(\ell - 1) + i, \end{aligned}$$

as desired. \square

Let \mathcal{S}_{2n}^k denote the set of symmetric $(2n)$ -cores with first part k , and Φ_n^k , the map Φ_n when the domain is restricted to \mathcal{S}_{2n}^k .

Corollary 5.4. *Under the bijection between symmetric $(2n)$ -cores and balanced flush abaci, $F_{\mathcal{A}}(\mathcal{S}_{2n}^k)$ is the set of abaci where the largest runner is located at level $\ell = \lceil \frac{k}{2n} \rceil$ of runner i , where $i := k \pmod{2n}$.*

Proposition 5.5. *The image of Φ_n^k is a subset of $\mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$, the set of $(2n-2)$ -cores with first part at most $k - \lceil \frac{k}{n} \rceil$.*

Proof. We may write k uniquely as $2n(\ell-1) + i$, where $\ell \geq 1$ and $1 \leq i \leq 2n$. By Lemma 5.3, elements of \mathcal{S}_{2n}^k correspond to balanced flush abaci where the largest active beads are located at level ℓ of the i th runner.

To show that the image of the map Φ_n^k is contained in $\mathcal{S}_{2n-2}^{\leq k - \lceil \frac{k}{n} \rceil}$, notice that the number of gaps in runner $2n+1-i$ that is smaller than the largest active bead (which is located in the i th runner) is equal to $\lceil \frac{k}{n} \rceil$. Indeed, it is equal to $2\ell-1$ if $i \equiv 1, \dots, n \pmod{2n}$, and 2ℓ if $i \equiv n+1, \dots, 2n \pmod{2n}$. When we apply the map Φ_n^k to this abacus, the number of gaps that are less than the first active bead decreases by at least $\lceil \frac{k}{n} \rceil$. Hence, the image of Φ_n^k is a subset of $\mathcal{S}_{2n-2}^{\leq k - \lceil \frac{k}{n} \rceil}$. \square

Recall that there is a bijective correspondence between elements in \tilde{C}_n/C_n and lattice points of the form $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Using the bijection in Hanusa-Jones [10], the lattice point (a_1, \dots, a_n) corresponds to the abacus with $2n$ runners, with the largest active bead for each runner located at levels $(a_1, \dots, a_n, -a_n, \dots, -a_1)$.

Fix an integer $k > 0$. Let $\ell_1 := k \pmod{n}$ and $\ell_2 := k \pmod{2n}$, where $1 \leq \ell_1 \leq n$ and $1 \leq \ell_2 \leq 2n$. Let H_n^k denote the affine hyperplane

$$H_n^k = \begin{cases} \{v \in \mathbb{R}^n : \langle v, \varepsilon_{\ell_1} \rangle = \lceil \frac{k}{2n} \rceil\} & \text{if } \ell_2 \in \{1, \dots, n\} \\ \{v \in \mathbb{R}^n : \langle v, \varepsilon_{n-\ell_1+1} \rangle = -\lceil \frac{k}{2n} \rceil\} & \text{if } \ell_2 \in \{n+1, \dots, 2n\} \end{cases}$$

Note that all of the points on H_n^k have the same fixed ℓ_1 th or $n-\ell_1+1$ th coordinate. In particular, for the first case, the ℓ_1 th coordinate is $\lceil \frac{k}{2n} \rceil$, and for the second case, it is $-\lceil \frac{k}{2n} \rceil$.

Proposition 5.6 (Elements of \mathcal{S}_{2n}^k lie on a hyperplane). *Under the correspondence between symmetric $(2n)$ -cores and coroot lattice points, we have the following:*

(i) *If $\ell_2 \in \{1, \dots, n\}$, then the symmetric $(2n)$ -cores λ with $\lambda_1 = k$ correspond to the lattice points $(a_1, \dots, a_n) \in H_n^k \cap \mathbb{Z}^n$ subject to the conditions*

$$\begin{aligned} -\left\lceil \frac{k}{2n} \right\rceil &< a_i \leq \left\lceil \frac{k}{2n} \right\rceil, & i \in [1, \ell_1 - 1], \\ -\left\lceil \frac{k}{2n} \right\rceil &< a_i < \left\lceil \frac{k}{2n} \right\rceil, & i \in [\ell_1 + 1, n]. \end{aligned}$$

(ii) *If $\ell_2 \in \{n+1, \dots, 2n\}$, then the symmetric $(2n)$ -cores λ with $\lambda_1 = k$ correspond to the lattice points $(a_1, \dots, a_k) \in H_n^k \cap \mathbb{Z}^n$ subject to the conditions*

$$\begin{aligned} -\left\lceil \frac{k}{2n} \right\rceil &< a_i \leq \left\lceil \frac{k}{2n} \right\rceil, & i \in [1, n - \ell_1], \\ -\left\lceil \frac{k}{2n} \right\rceil &\leq a_i \leq \left\lceil \frac{k}{2n} \right\rceil, & i \in [n - \ell_1 + 2, n]. \end{aligned}$$

Proof. The claim follows from Corollary 5.4. \square

Proposition 5.7. *Under the correspondence between $(2n-2)$ -cores and lattice points in \mathbb{Z}^{n-1} , we have the following:*

(i) If $\ell_2 \in \{1, \dots, n\}$, then elements of $\mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$ correspond to lattice points $(a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$ subject to the conditions

$$\begin{aligned} -\left\lceil \frac{k}{2n} \right\rceil < a_i &\leq \left\lceil \frac{k}{2n} \right\rceil, & i \in [1, \ell_1 - 1], \\ -\left\lceil \frac{k}{2n} \right\rceil < a_i &< \left\lceil \frac{k}{2n} \right\rceil, & i \in [\ell_1, n - 1]. \end{aligned}$$

(ii) If $\ell_2 \in \{n+1, \dots, 2n\}$, then elements of $\mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$ correspond to lattice points $(a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$ subject to the conditions

$$\begin{aligned} -\left\lceil \frac{k}{2n} \right\rceil < a_i &\leq \left\lceil \frac{k}{2n} \right\rceil, & i \in [1, n - \ell_1], \\ -\left\lceil \frac{k}{2n} \right\rceil \leq a_i &\leq \left\lceil \frac{k}{2n} \right\rceil, & i \in [n - \ell_1 + 1, n]. \end{aligned}$$

Proof. We will prove the claim for case (i), when $1 \leq \ell_2 \leq n$. The second case is proved analogously. The coroot lattice points satisfying condition (i) above, correspond to all balanced flush abaci whose highest active beads are no higher than the bead located at level $\lceil \frac{k}{2n} \rceil$ of runner $\ell_1 - 1$.

Since $1 \leq \ell_2 \leq n$, we may write $k = 2n\ell + \ell_1$. It follows from Lemma 5.3 that the first part of the core partition is at most

$$\begin{aligned} (2n-2) \left(\left\lceil \frac{k}{2n} - 1 \right\rceil \right) + \ell_1 - 1 &= (2n-2)\ell + \ell_1 - 1 \\ &= (2n\ell + \ell_1) - (2\ell + 1) \\ &= k - \left\lceil \frac{k}{n} \right\rceil, \end{aligned}$$

as desired. \square

Denote the set of coroot lattice points corresponding to the core partitions \mathcal{S}_{2n}^k as \mathcal{R}_{2n}^k , and the set of coroot lattice points corresponding to the core partitions $\mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$ as $\mathcal{R}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$. We are now ready to prove the following theorem:

Theorem 5.8. *The map*

$$\Phi_n^k : \mathcal{S}_{2n}^k \rightarrow \mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$$

is a bijection.

Proof. Using the commutative diagram 5.1 and the bijection $F_{\mathcal{R}}$ between abacus diagrams and coroot lattice points, the following diagram is commutative:

$$(5.2) \quad \begin{array}{ccc} \mathcal{S}_{2n}^k & \xrightarrow{F_{\mathcal{R}} \circ F_{\mathcal{A}}} & \mathcal{R}_{2n}^k \\ \downarrow \Phi_n^k & & \downarrow \Phi_n^k \\ \mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)} & \xrightarrow{F_{\mathcal{R}} \circ F_{\mathcal{A}}} & \mathcal{R}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)} \end{array}$$

In the commutative diagram above, Φ_n^k acts on the coroot lattice points in \mathcal{R}_{2n}^k by deleting the (ℓ_1) th coordinate if $1 \leq \ell_2 \leq n$, or deleting the $(n - \ell_1 + 1)$ th coordinate if $n + 1 \leq \ell_2 \leq n$. This is clearly an injection because in each case, the ℓ_1 th and $(n - \ell_1 + 1)$ th coordinates are both fixed and redundant.

The map Φ_n^k is a bijection between \mathcal{R}_{2n}^k and $\mathcal{R}_{2n-2}^{\leq(k-\lceil \frac{k}{2n} \rceil)}$ because the inverse is defined by inserting $\lceil \frac{k}{2n} \rceil$ at position ℓ_1 if $1 \leq \ell_2 \leq n$, or $-\lceil \frac{k}{2n} \rceil$ at position $n - \ell_1 + 1$ if $n + 1 \leq \ell_2 \leq 2n$. Since the horizontal arrows in the commutative diagram are also bijections, the map $\Phi_n^k : \mathcal{S}_{2n}^k \rightarrow \mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{2n} \rceil)}$ is a bijection, as desired. \square

6. GEOMETRIC INTERPRETATION OF THE MAP Φ_n^k

In this section we describe a geometric interpretation of Φ_n^k on \tilde{C}_n/C_n using the alcove model. Recall the correspondence between \tilde{C}_n and the alcoves in \mathbb{R}^n given by sending a word $w \in \tilde{C}_n$ to $w(\mathcal{A}_o)$. Throughout this section we shall only consider the case when $\ell_2 \in \{1, \dots, n\}$ since the proof for when $\ell_2 \in \{n+1, \dots, 2n\}$ is completely analogous. Moreover, ℓ_1 will be denoted by ℓ for the rest of this section.

We begin by noting that we can naturally identify H_n^k with \mathbb{R}^{n-1} via the $\{\varepsilon_i : i \neq \ell\}$ basis where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the standard orthonormal basis for \mathbb{R}^n . We define the map $\pi : \mathbb{R}^n \rightarrow H_n^k$ to be the projection of \mathbb{R}^n onto H_n^k . Analytically, this is given by

$$(6.1) \quad \pi(v) = \sum_{j \neq \ell} \langle v, \varepsilon_j \rangle \varepsilon_j + \left\lceil \frac{k}{2n} \right\rceil \varepsilon_\ell.$$

For the duration of this paper we fix an n and a k so that we only have to deal with the projection π_n^k , which will now be denoted by π . Moreover, whenever we are dealing with an object which is naturally indexed by n and k , even if no label is explicitly mentioned, we will assume this fixed labeling.

We now define some important terminology. We call two hyperplanes H_α and H_β *parallel* (resp. *perpendicular*) if α and β are parallel (resp. perpendicular). A *step* is the result of a flipping of an alcove over a hyperplane which bounds the alcove. We call a step from an alcove \mathcal{A} to an alcove \mathcal{A}' *perpendicular* if it is achieved by reflecting \mathcal{A} over a hyperplane not perpendicular to H_n^k . The reason for this terminology is clear as illustrated by Figure 10. A step that is not perpendicular shall be called *parallel*.

Suppose that

$$\mathcal{A}_o = \mathcal{A}^1 \rightarrow \dots \rightarrow \mathcal{A}^r = \mathcal{A}_w$$

is a minimal length alcove walk from \mathcal{A}_o to \mathcal{A}_w . Now we aim to show that

$$(6.2) \quad \pi(\mathcal{A}^1) \rightarrow \dots \rightarrow \pi(\mathcal{A}^r)$$

is an alcove walk for $\mathcal{A}_{\Phi_n^k(w)}$ and that if one removes all repeated instances of alcoves in (6.2), then the resulting walk is minimal.

We begin by showing that

$$(6.3) \quad \pi(\mathcal{A}^1) \rightarrow \dots \rightarrow \pi(\mathcal{A}^r)$$

is indeed an alcove walk for $\Phi_n^k(w)$. It is sufficient to show that the image of an alcove $\mathcal{A} \subseteq \mathbb{R}^n$ under π is an alcove when we identify H_n^k with \mathbb{R}^{n-1} .

Lemma 6.1. *The image of an alcove under π is an alcove via the identification of H_n^k with \mathbb{R}^{n-1} .*

Proof. Let $\varepsilon_1, \dots, \varepsilon_n$ be the standard basis of \mathbb{R}^n . Recall that the positive roots of the affine Weyl group \tilde{C}_n are $2\varepsilon_i$, $1 \leq i \leq n$, and $\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$. Without loss of generality, we may assume that $\ell = 1$, so that the hyperplane H_n^k is parallel to the hyperplane $H_{\varepsilon_1} = \{\lambda \in \mathbb{R}^n, (\lambda, 2\varepsilon_1) = 0\}$.

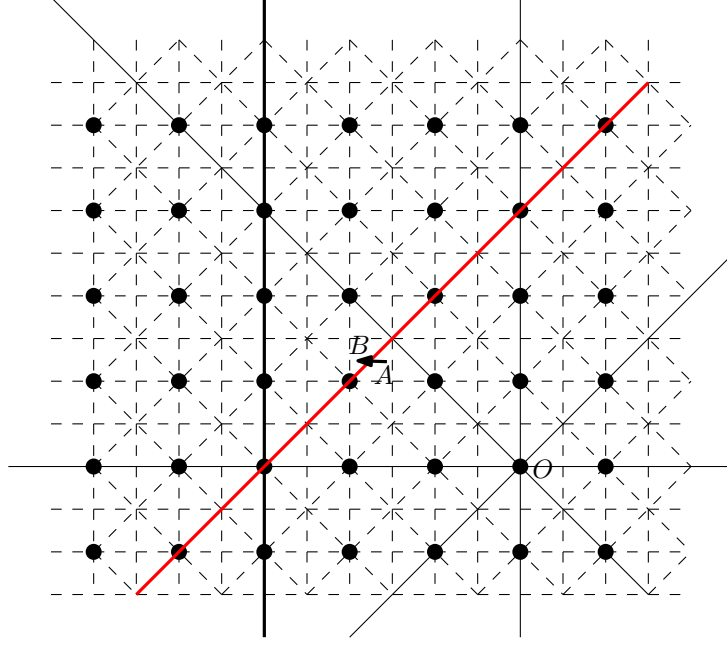


FIGURE 10. A perpendicular step (black arrow) is obtained by reflecting over a hyperplane (red line) not perpendicular to H_n^k .

Fix an alcove \mathcal{A} . For every positive root r , there exists a unique integer k_r such that $\lambda \in \mathbb{R}^n$ lies in the interior of the alcove \mathcal{A} if and only if λ satisfies the inequalities $k_r < (\lambda, r) < k_r + 1$ for all positive roots r (i.e. λ lies between the hyperplanes H_{r,k_r} and H_{r,k_r+1}).

In particular, by page 91 in [11], we have a system of simple roots Δ , which is the set of roots $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$, and $\alpha_n = 2\varepsilon_n$. The unique long root is $\tilde{\alpha} = 2\varepsilon_1$. The fundamental alcove is the region bounded by the hyperplanes $H_{\alpha_i,0}$ for $1 \leq i \leq n$, and $H_{\tilde{\alpha},1}$. In other words, the fundamental alcove consists of all points $\lambda \in \mathbb{R}^n$ such that $(\lambda, \alpha_i) > 0$ for $1 \leq i \leq n$ and $(\lambda, \tilde{\alpha}) < 1$. This is precisely all points (a_1, \dots, a_n) satisfying the inequality

$$\frac{1}{2} > a_1 > a_2 > \dots > a_n > 0.$$

Now, the alcove \mathcal{A} is a translation by integer coordinates of an alcove in the fundamental region. To show that the image of \mathcal{A} is an alcove, it suffices to show that the image of any alcove in the fundamental region is an alcove. All the alcoves in the fundamental region are obtained from the fundamental alcove via a sequence of reflections across the hyperplanes $H_{r,0}$, where $r \in \Phi^+$ is a positive root.

There are three types of reflections to consider:

- (i) Reflecting across the hyperplane $H_{\varepsilon_i - \varepsilon_j, 0}$. In this case, we switch the i th coordinate and the j th coordinate, i.e.,

$$(a_1, \dots, a_i, \dots, a_j, \dots, a_n) \mapsto (a_1, \dots, a_j, \dots, a_i, \dots, a_n).$$

- (ii) Reflecting across the hyperplane $H_{\varepsilon_i + \varepsilon_j, 0}$. In this case, we first switch the i th coordinate and the j th coordinate, then also change their signs, i.e.,

$$(a_1, \dots, a_i, \dots, a_j, \dots, a_n) \mapsto (a_1, \dots, -a_j, \dots, -a_i, \dots, a_n).$$

(iii) Reflecting across the hyperplane $H_{2\varepsilon_i,0}$. In this case, we only change the sign of the i th coordinate, i.e.,

$$(a_1, \dots, a_i, \dots, a_n) \mapsto (a_1, \dots, -a_i, \dots, a_n).$$

It is easy to see from the three cases above that an alcove in the fundamental region consists of all points $(k_1 a_{g(1)}, \dots, k_n a_{g(n)}) \in \mathbb{R}^n$ such that

$$\frac{1}{2} > a_1 > a_2 > \dots > a_n > 0,$$

where $k_i \in \{-1, 1\}$ for $1 \leq i \leq n$, and g is a permutation of $\{1, 2, \dots, n\}$.

To finish the proof, we note that by applying the projection, the image contains all points of the form $(k_2 a_{g(2)}, \dots, k_n a_{g(n)}) \in \mathbb{R}^{n-1}$ such that

$$\frac{1}{2} > a_1 > a_2 > \dots > a_n > 0.$$

Since $a_{g(1)}$ is gone, we can just delete it from the inequality. It follows that the image is an alcove of \tilde{C}_{n-1}/C_{n-1} , as desired. \square

Remark 6.2. By considering the image of an alcove under the projection π from (6.1), Lemma 6.1 tells us that π induces a map $\tilde{C}_n \rightarrow \tilde{C}_{n-1}$. By abuse of notation, we denote this induced map by π as well.

Lemma 6.3. *The image of a fundamental domain under π is a fundamental domain via the identification of H_n^k with \mathbb{R}^{n-1} .*

Proof. The fundamental domain in \mathbb{R}^{n-1} is the region containing the points (a_1, \dots, a_n) where $-\frac{1}{2} \leq a_i \leq \frac{1}{2}$ for all $1 \leq i \leq n$. Without loss of generality, we may assume $\ell = 1$, so that applying π deletes the first coordinate. The image of the fundamental domain is (a_2, \dots, a_n) , where $-\frac{1}{2} \leq a_i \leq \frac{1}{2}$ for all $2 \leq i \leq n$. This is clearly the fundamental domain in \mathbb{R}^{n-1} , as desired. \square

Definition 6.4. Consider the alcoves whose coroot lattice points are in \mathcal{R}_{2n}^k . Define the set of *good alcoves* to be the subset of these alcoves which share a face with the hyperplane $T_n^k = \{v \in \mathbb{R}^n : \langle v, \varepsilon_\ell \rangle = \lceil \frac{k}{2n} \rceil - \frac{1}{2}\}$.

Also, from now on denote by \mathcal{A}'_\circ the distinguished alcove whose coroot lattice point is $\lceil \frac{k}{2n} \rceil \varepsilon_\ell$.

Proposition 6.5. *Distinguished alcoves whose coroot lattice points are in \mathcal{R}_{2n}^k are good.*

Proof. There exists a minimal length alcove walk from \mathcal{A}_\circ to \mathcal{A}'_\circ that is a straight line perpendicular to H_n^k . After this alcove walk passes through the hyperplane T_n^k , it enters the region corresponding to the coset with coroot lattice point $\lceil \frac{k}{2n} \rceil \varepsilon_\ell$. It is clear that this initially hit alcove must be distinguished. Hence, it follows that this alcove is \mathcal{A}'_\circ and that \mathcal{A}'_\circ is good.

The coroot lattice points lying in \mathcal{R}_{2n}^k can be partitioned into $(2n - 2)$ Weyl chambers. Within each Weyl chamber, the positions of the distinguished alcoves within each fundamental region translates are the same. Therefore, the distinguished alcoves in the Weyl chamber containing the lattice point $\lceil \frac{k}{2n} \rceil \varepsilon_\ell$ are all good. Within a Weyl chamber every distinguished alcove is a translation from a fixed alcove by a lattice point parallel to H_n^k . And, since translation by such lattice points doesn't affect goodness it follows that if one distinguished alcove in a given chamber is good, then so are the rest of these alcoves.

Now there exists, in each Weyl chamber, a representative which is obtained by applying to \mathcal{A}'_\circ a sequence of perpendicular reflections to ε_ℓ . Since such reflections preserve goodness, there exists a good representative in each Weyl chamber. By the previous comment, the theorem follows, as desired. \square

Lemma 6.6. *There are exactly k hyperplanes lying on the minimal alcove walk from \mathcal{A}_o to \mathcal{A}'_o .*

Proof. Note the coroot lattice point associated with \mathcal{A}'_o is $\lceil \frac{k}{2n} \rceil \varepsilon_\ell$. It suffices to find the length of the reduced word corresponding to \mathcal{A}'_o . The abacus corresponding to this point has the lowest bead at level $\lceil \frac{k}{2n} \rceil$ in runner ℓ , at level $-\lceil \frac{k}{2n} \rceil$ in runner $N - \ell$, and at level zero elsewhere. We see that the value of the lowest bead B in runner ℓ is $N \lceil \frac{k}{2n} \rceil + \ell - 1$. Furthermore, there exists a unique bead b on runner ℓ whose value lies in the range $[n+1, N+n]$. Note that the value labeling b is ℓ if $\ell \geq n+1$ and $\ell + N$ otherwise. If g denotes the number of gaps between B and b , and p denotes the number of beads whose values are greater than $N+n$, then

$$(6.4) \quad g = \begin{cases} (2n-1) \left(\lceil \frac{k}{2n} \rceil - 1 \right) + \ell - 1 & \text{if } \ell \geq n+1, \\ (2n-1) \left(\lceil \frac{k}{2n} \rceil - 1 \right) & \text{if } \ell < n+1, \end{cases}$$

and

$$(6.5) \quad p = \begin{cases} \lceil \frac{k}{2n} \rceil & \text{if } \ell \geq n+1, \\ \lceil \frac{k}{2n} \rceil - 1 & \text{if } \ell < n+1. \end{cases}$$

Using Corollary 8.1 in [10], we know then that the length of the word corresponding to \mathcal{A}'_o is

$$(6.6) \quad \begin{cases} g + p = (2n-1) \left(\lceil \frac{k}{2n} \rceil - 1 \right) + \ell - 1 + \lceil \frac{k}{2n} \rceil & \text{if } \ell \geq n+1, \\ g + p + (b - N) = (2n-1) \left(\lceil \frac{k}{2n} \rceil - 1 \right) + \lceil \frac{k}{2n} \rceil - 1 + (\ell + N) - N & \text{if } \ell < n+1. \end{cases}$$

Both of these are equal to k because by the definition of ℓ , $k = 2n(\lceil \frac{k}{2n} \rceil - 1) + \ell$. From this we conclude that the minimum length path from \mathcal{A}_o to \mathcal{A}'_o passes through precisely k hyperplanes. \square

Proposition 6.7. *Any minimal length alcove walk to a good alcove takes exactly k perpendicular steps to H_n^k .*

Proof. Observe from the symmetry of the hyperplane arrangement that steps in the walk that are parallel to H_n^k preserve the number of perpendicular steps remaining to T_n^k . The proof follows easily from this observation and Lemma 6.6. \square

Lemma 6.8 (Walk Lifting Lemma). *Let \mathcal{A} be a good alcove. If the minimal length alcove walk from $\pi(\mathcal{A}_o)$ to $\pi(\mathcal{A})$ has length s , then any minimal length alcove walk from \mathcal{A}_o to \mathcal{A} has length $k + s$.*

Proof. For any minimal length alcove walk from \mathcal{A}_o to \mathcal{A} of length ℓ , consider its projection to the hyperplane T_n^k , the hyperplane parallel to H_n^k which contains a face of \mathcal{A} , as described in Definition 6.4. This projection deletes all the perpendicular steps in the walk that is perpendicular to H_n^k . By Proposition 6.7, the projection is an alcove walk from $\pi(\mathcal{A}_o) \rightarrow \pi(\mathcal{A})$ of length $\ell - k$. Since the minimal length alcove walk from $\pi(\mathcal{A}_o) \rightarrow \pi(\mathcal{A})$ has length s , $\ell - k \geq s$, or equivalently $\ell \geq k + s$. To prove $\ell = k + s$, it suffices to show that there exists a minimal length alcove walk from $\pi(\mathcal{A}_o)$ to $\pi(\mathcal{A})$ that can be lifted to a path from \mathcal{A}_o to \mathcal{A} , containing only forward perpendicular steps and parallel steps to H_n^k (as long as the walk from \mathcal{A}_o to \mathcal{A} contains only forward perpendicular steps and parallel steps, it will contain exactly k perpendicular steps to H_n^k).

We can index hyperplanes T_n^k by levels, based on their distance from the origin. We will prove the claim by using induction on the level of the hyperplane in which the good alcove \mathcal{A} is located.

We first consider the case where $r = 1$, which consists of good alcoves whose corresponding coroot lattice point is at most two in each coordinate. We just need to show there is a lifted path from \mathcal{A}_o to any good alcove at level 1 which takes only parallel and forward perpendicular steps.

Observe that each alcove in \tilde{C}_n is contained in a set of alcoves which project down to a translate of $\mathcal{B}_{o,n-1}$; we call such a set of alcoves a cluster. All alcoves in a cluster are reachable from each other via a path which takes only parallel steps. These clusters are centered at either a coroot lattice

point or a coroot lattice point translated by $(1/2, \dots, 1/2)$. This allows movement parallel to H_n^k up to a unit in each direction without taking a perpendicular step. Since all alcoves in a cluster have a face which lies in a hyperplane parallel to H_n^k , between any two hyperplanes parallel to H_n^k we can make exactly one forward perpendicular step across a diagonal hyperplane, switching in which of the two hyperplanes the alcoves' faces lie.

Consider a good alcove \mathcal{A} on level 1. Since it's on level 1, \mathcal{A} 's coroot lattice point is between -1 and 1 in each coordinate, and so either $\pi(\mathcal{A}) = \pi(\mathcal{A}')$ for some alcove \mathcal{A}' in the fundamental region, or \mathcal{A} lies outside the fundamental region by at most one unit in each direction. In the first case, it's clear \mathcal{A} is reachable via only parallel and forward perpendicular steps. In the second, in moving from the fundamental region to level 1 toward H_n^k , the path passes through enough clusters to allow the parallel movement required to reach \mathcal{A} . In either case, it is possible to lift a path to any good alcove on level 1.

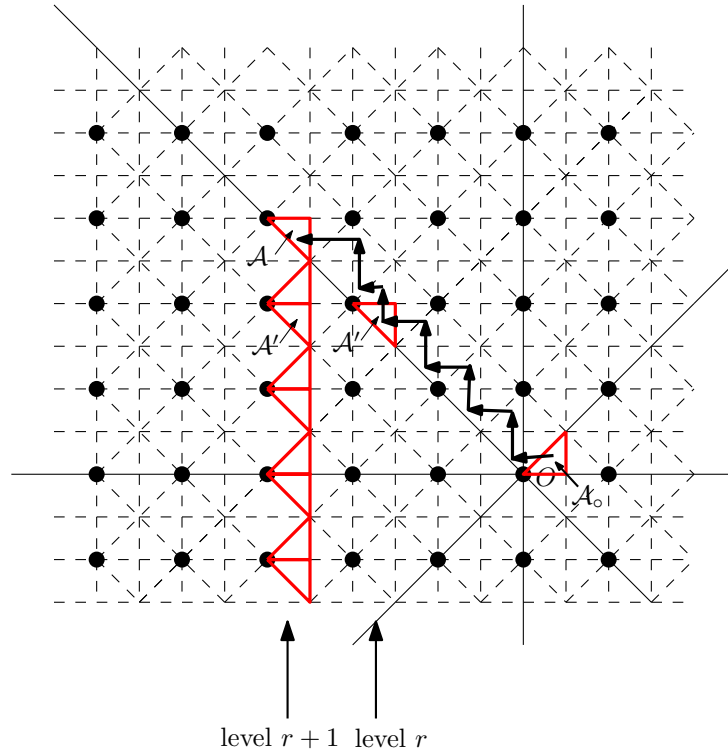


FIGURE 11. Going from an good alcove at level r to an good alcove at level $r + 1$.

For the induction hypothesis, suppose at level r , all the alcoves satisfy the walk lifting lemma. So for each good alcove at level k , there exists a shortest path from $\pi(\mathcal{A}_0)$ to that alcove that can be lifted. Now, consider a good alcove \mathcal{A} , located at level $r + 1$. If this good alcove comes from the projection of a good alcove located at level r , then we can just take the minimal alcove walk from $\pi(\mathcal{A}_0) \rightarrow \pi(\mathcal{A})$ to be the same as the minimal alcove walk at level r . The shortest path at level r can be lifted to a shortest path from $\mathcal{A}_0 \rightarrow \pi_r(\mathcal{A})$. Once we reach $\pi_r(\mathcal{A})$, we can make a sequence of perpendicular steps to level $r + 1$ to obtain a minimal alcove walk from \mathcal{A}_0 to \mathcal{A} .

Now, suppose the good alcove at level $r + 1$ is not the projection of an alcove at level r . Consider a minimal alcove walk from $\pi(\mathcal{A}_0)$ to $\pi(\mathcal{A})$. Let \mathcal{A}' be the last good alcove in this walk that comes from the projection of a good alcove at level r , which by an abuse of notation we will call it \mathcal{A}' .

We can replace the path from $\pi(\mathcal{A}_o)$ to \mathcal{A}' by a minimal alcove walk at level r that can be lifted, by the induction hypothesis. Now we just have to show that after we reach the pre-image of \mathcal{A}' at level r , we can go to \mathcal{A} by only using forward perpendicular steps and parallel steps. This is true because we have essentially expanded the boundary by 1 in each dimension by going down from level r to level $r + 1$, and the base case of the induction guarantees such a path (see Figure 11). This proves the claim for level $r + 1$. By induction, it follows that all the good alcoves satisfy the walk lifting lemma, as desired. \square

Lemma 6.9. *For any alcove $\mathcal{B} \in \mathbb{R}^{n-1}$ whose coroot lattice point is in $\mathcal{R}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$, there exists a good alcove \mathcal{A} such that $\pi(\mathcal{A}) = \mathcal{B}$.*

Proof. Let $y \in \mathcal{R}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)}$ be the coroot lattice point for the alcove \mathcal{B} . By Proposition 5.6 and Proposition 5.7, there exists a coroot lattice point $y' \in \mathcal{R}_{2n}^k$ such that $\pi(y') = y$. Furthermore, from Lemma 6.3, we know that the translated fundamental domain centered at y' is mapped to the translated fundamental domain centered at $\pi(y') = y$. It suffices to show that there exists a good alcove with coroot lattice point y' whose image under π is \mathcal{B} .

By Lemma 6.1, the image of any good alcove with coroot lattice point y' is an alcove with coroot lattice point y . Moreover, it is clear that the image under π of the intersection of T_n^k and the translated fundamental domain centered around y' is the translated fundamental domain centered around y . It follows that there exists a good alcove \mathcal{A} such that $\pi(\mathcal{A}) = \mathcal{B}$, as desired. \square

Let \mathcal{X}_{2n}^k denote the distinguished alcoves whose associated lattice points are elements of \mathcal{R}_{2n}^k and let $\mathcal{X}_{2n}^{\leq(k-\lceil \frac{k}{n} \rceil)}$ be defined similarly.

Theorem 6.10. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{S}_{2n}^k & \xrightarrow{F\mathcal{X}} & \mathcal{X}_{2n}^k \\ \downarrow \Phi_n^k & & \downarrow \pi \\ \mathcal{S}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)} & \xrightarrow{F\mathcal{X}} & \mathcal{X}_{2n-2}^{\leq(k-\lceil \frac{k}{n} \rceil)} \end{array}$$

Proof. Start with a core $S \in \mathcal{S}_{2n}^k$. By the commutative diagram (5.2), the coroot lattice point associated to the image of $F\mathcal{X}(S)$ is precisely the same as the coroot lattice point associated to the alcove $F\mathcal{X}(\Phi_n^k(S))$. To complete the theorem, it suffices to show that π sends distinguished alcoves to distinguished alcoves.

To see this, suppose that $\mathcal{A} \in \mathcal{X}_{2n}^k$ is an alcove such that $\pi(\mathcal{A})$ is not distinguished. Let the minimal length alcove walk to $\pi(\mathcal{A})$ be s . There exists an alcove \mathcal{B} with the same coroot lattice point as $\pi(\mathcal{A})$ which has a minimal length walk of length $r < s$. By Lemma 6.9, there exists a good alcove \mathcal{A}' in the same coset as \mathcal{A} so that $\pi(\mathcal{A}') = \mathcal{B}$. Since both \mathcal{A} and \mathcal{A}' are good, by the Walk Lifting Lemma there exist minimal length alcove walks to \mathcal{A} and \mathcal{A}' of lengths $k + s$ and $k + r$, respectively. Since $k + s > k + r$, this is a contradiction to the fact that \mathcal{A} is a distinguished alcove. It follows that $\pi(\mathcal{A})$ must be distinguished, as desired. \square

We may now conclude that $\pi(\mathcal{A}_o)$ is the fundamental alcove for \tilde{C}_{n-1} and that $\pi(\mathcal{A}_w)$ corresponds to $\Phi_n^k(w)$. Moreover, π takes adjacent alcoves to adjacent alcoves, so (6.3) is an alcove walk from the identity alcove to the alcove corresponding to $\Phi_n^k(w)$.

Theorem 6.11. *Let w be a minimal length coset representative for \tilde{C}_n/C_n such that \mathcal{C}_w has first part k . If*

$$(6.7) \quad \mathcal{A}^1 \rightarrow \cdots \rightarrow \mathcal{A}^r$$

is an alcove walk for w , then

$$(6.8) \quad \pi(\mathcal{A}^1) \rightarrow \cdots \rightarrow \pi(\mathcal{A}^r)$$

is an alcove walk for $\Phi_n^k(w)$. Moreover, if one removes all repeated instances of alcoves in (6.8), then the resulting walk is a minimal length alcove walk for $\Phi_n^k(w)$.

Proof. By Lemma 6.1, Theorem 6.10, and the above remarks, it's clear that (6.8) does in fact give an alcove walk for $\Phi_n^k(w)$. The removal of the repeated instances of alcoves still yields an appropriate alcove walk under π . Moreover, we see by Lemma 6.8 (Walk Lifting Lemma) that this must in fact be a minimal walk, as any shorter walk would correspond to a shorter original walk to \mathcal{A}_w , which is a contradiction. \square

Corollary 6.12. *Let w be a minimum length coset representative for \tilde{C}_n/C_n . Then*

$$\ell_{\tilde{C}_n}(w) - \ell_{\tilde{C}_{n-1}}(\Phi_n^k(w)) = k.$$

Proof. Start with a minimal length alcove walk

$$\mathcal{A}_o = \mathcal{A}^1 \rightarrow \cdots \rightarrow \mathcal{A}^{s+1} = F_{\mathcal{X}}(w),$$

where $s = \ell_{\tilde{C}_n}(w)$. Then

$$\pi(\mathcal{A}_o) \rightarrow \cdots \rightarrow \pi(F_{\mathcal{X}}(w))$$

is an alcove walk for $F_{\mathcal{X}}(\Phi_n^k(w))$. Note that in this alcove walk there are exactly k repeated instances of alcoves, corresponding to the k perpendicular steps to H_n^k . We may delete these repeated instances of alcoves to get an alcove walk of length $s - k$. This alcove walk is minimal, for if it is not, then there exists an alcove walk of length $r < s - k$, which, by Lemma 6.8 (Walk Lifting Lemma), may be lifted to an alcove walk from \mathcal{A}_o to $F_{\mathcal{X}}(w)$ of length $r + k < s$. This is a contradiction. Therefore $\ell_{\tilde{C}_{n-1}}(\Phi_n^k(w)) = \ell_{\tilde{C}_n}(w) - k$ and the claim follows, as desired. \square

7. ACTION OF Φ_n^k ON REDUCED WORDS

In this section, we will describe the action of the map Φ_n^k on reduced words of \tilde{C}_n/C_n . Let $s_{i_1} \cdots s_{i_\ell}$ be a reduced word in \tilde{C}_n/C_n which corresponds to the abacus $\mathbf{a} = (a_1, \dots, a_n, -a_n, \dots, -a_1)$. From the correspondence between reduced words and abacus diagrams, we can see that $s_{i_\ell} \cdots s_{i_1}(\mathbf{a}) = (0, \dots, 0)$, is the abacus diagram corresponding to the identity element in \tilde{C}_n/C_n .

From $s_{i_1} \cdots s_{i_\ell}$, we can build a reduced word for $\Phi_n^k(\mathbf{a})$ in ℓ steps: first, consider the action of s_{i_1} on the abacus \mathbf{a} . Using the results of Section 3.2 in [10], if s_{i_1} acts on \mathbf{a} by changing the position of the largest runner, set $t_1 = 1 \in \tilde{C}_{n-1}$. Otherwise, if applying s_{i_1} to \mathbf{a} does not change the position of the largest runner, then there exists a unique generator $s_{i'_1} \in \tilde{C}_{n-1}$, where $i'_1 = i_1$ or $i_1 - 1$, depending on the positions of the runners being changed relative to the largest runner, such that $\Phi_n(s_{i_1}\mathbf{a}) = s_{i'_1}\Phi_n^k(\mathbf{a})$. In this case, set $t_1 = s_{i'_1}$.

At step r , $1 \leq r \leq \ell$, consider the action of s_{i_r} on the abacus $s_{i_{r-1}} \cdots s_{i_1} \cdot \mathbf{a}$. If s_{i_r} changes the position of the largest runner, set $t_r = 1 \in \tilde{C}_{n-1}$. Otherwise, there exists a unique generator $s_{i'_r} \in \tilde{C}_{n-1}$ such that

$$\Phi_n(s_{i_r}s_{i_{r-1}} \cdots s_{i_1} \cdot \mathbf{a}) = s_{i'_r}\Phi_n(s_{i_{r-1}} \cdots s_{i_1} \cdot \mathbf{a}) = s_{i'_r}t_{r-1} \cdots t_1 \cdot \Phi_n(\mathbf{a}).$$

In this case, set $t_r = s_{i'_r}$.

From our construction, we obtain the following commutative diagram for all $1 \leq r \leq \ell$:

$$\begin{array}{ccc} s_{i_{r-1}} \cdots s_{i_1} \cdot \mathbf{a} & \xrightarrow{\Phi_n} & t_{r-1} \cdots t_1 \cdot \Phi_n(\mathbf{a}) \\ \downarrow s_{i_r} & & \downarrow t_r \\ s_{i_r} s_{i_{r-1}} \cdots s_{i_1} \cdot \mathbf{a} & \xrightarrow{\Phi_n} & t_r t_{r-1} \cdots t_1 \cdot \Phi_n(\mathbf{a}) \end{array}$$

It follows that $t_1 \cdots t_\ell$ is a word corresponding to the abacus $\Phi_n(\mathbf{a})$. By Lemma 5.3, since $s_{i_1} \cdots s_{i_\ell}$ is a reduced word for the abacus \mathbf{a} , there are exactly λ_1 instances when the generator s_{i_j} changes the largest runner. In fact, it will either move the largest runner to the left by one, or, if the largest runner is the first runner, decrease the level of the largest runner by one and move it to runner $2n$. In other words, the word $t_1 \cdots t_\ell$ has length of exactly $\ell - \lambda_1$.

Proposition 7.1. *The word $t_1 \cdots t_\ell$ is a reduced word corresponding to the abacus $\Phi_n(\mathbf{a})$.*

Proof. It suffices to show that the length of a reduced word for $\Phi_n(\mathbf{a})$ is $\ell - \lambda_1$. We will proceed via contradiction. Suppose $s_{i_1} \cdots s_{i_p}$ is a reduced word for $\Phi_n(\mathbf{a})$, with $p < \ell - \lambda_1$. We will construct a word for \mathbf{a} that has length $p + \lambda_1 < \ell$.

By definition, $s_{i_p} \cdots s_{i_1} \cdot \Phi_n(\mathbf{a}) = (0, \dots, 0)$, the abacus representing the identity element of the quotient \tilde{C}_{n-1}/C_{n-1} . We will construct a word for \mathbf{a} as follows. For the first step, consider the action of s_{i_1} on the abacus $\Phi_n(\mathbf{a})$. There are two cases to consider:

- (i) $1 \leq i_1 \leq n-1$. The generator s_{i_1} swaps two runners (along with their symmetric runners). Consider the positions of the two runners in the abacus \mathbf{a} . If they are adjacent in \mathbf{a} , then there exists a generator $s_{i'_1} \in \tilde{C}_n$ such that $s_{i'_1}$ switches these two runners in the abacus \mathbf{a} . If this is the case, set $w_1 = s_{i'_1}$. On the other hand, if the two runners are not adjacent in \mathbf{a} , then they must be separated by the longest runner or the symmetric runner of the longest runner. In this case, set $w_1 = s_{i''_1} s_{i'_1}$, where $s_{i'_1}$ is the generator that moves the longest runner to the left by one position, and $s_{i''_1}$ is the generator that swaps the two desired runners.
- (ii) $i_1 = 0$. The generator s_0 swaps the first runner and the last runner, increases the first runner by one level, and decreases the last runner by one level. Consider the position of this first runner in the abacus \mathbf{a} . If this runner is the first runner, set $w_1 = s_0$. On the other hand, if this runner is not the first runner, then the first runner must be the longest runner or the symmetric runner of the longest runner. If the first runner is the longest runner, set $w_1 = s_0 s_1 s_0$. If the first runner is the symmetric runner of the longest runner, then set $w_1 = s_0 s_1$.

It can be checked from our construction that $w_1 \mathbf{a} = s_{i_1} \Phi_n(\mathbf{a})$.

At step r , $1 \leq r \leq p$, suppose we have inductively constructed w_j for all $1 \leq j \leq r-1$ such that $\Phi_n(w_{r-1} \cdots w_1 \cdot \mathbf{a}) = s_{i_{r-1}} \cdots s_{i_1} \cdot \Phi_n(\mathbf{a})$. Consider the action of s_{i_r} on the abacus $s_{i_{r-1}} \cdots s_{i_1} \cdot \Phi_n(\mathbf{a})$. We construct w_r in the same as was described above so that the following diagram commutes:

$$\begin{array}{ccc} w_{r-1} \cdots w_1 \cdot \mathbf{a} & \xrightarrow{\Phi_n} & s_{i_{r-1}} \cdots s_{i_1} \cdot \Phi_n(\mathbf{a}) \\ \downarrow w_r & & \downarrow s_{i_r} \\ w_r w_{r-1} \cdots w_1 \cdot \mathbf{a} & \xrightarrow{\Phi_n} & s_{i_r} s_{i_{r-1}} \cdots s_{i_1} \cdot \Phi_n(\mathbf{a}) \end{array}$$

To finish our construction, notice that after step p , $\Phi_n(w_p w_{p-1} \cdots w_1 \cdot \mathbf{a}) = (0, \dots, 0)$ is the abacus corresponding to the identity element in \tilde{C}_{n-1}/C_{n-1} . However, $w_p w_{p-1} \cdots w_1 \cdot \mathbf{a}$ is not necessarily

the abacus corresponding to the identity element in \tilde{C}_n/C_n . Let w_{p+1} be the shortest word so that $w_{p+1}w_p \cdots w_1 \cdot \mathbf{a} = (0, \dots, 0)$ is the abacus corresponding to the identity element in \tilde{C}_n/C_n .

From our construction, the word $(w_{p+1} \cdots w_1)^{-1} = w_1^{-1} \cdots w_{p+1}^{-1}$ is a word corresponding to the abacus \mathbf{a} , and it has length $\ell(s_{i_1} \cdots s_{i_p}) + \lambda_1 = p + \lambda_1$. This is because every extra generator we added moves the largest runner to the left, or moves it from the first runner to the last runner while decreasing its level by one. Since $p + \lambda_1 < \ell$, the length of the word corresponding to the abacus \mathbf{a} , we have arrived at a contradiction. The length of $\Phi_n(\mathbf{a})$ is $\ell - \lambda_1$, as desired. \square

Example 7.2. The reduced word for the abacus $\mathbf{a} = (2, 1, -1, 1, -1, -2)$ is $s_0s_1s_3s_2s_3s_0s_1s_2s_0s_1s_0$. Using our procedure, a reduced word for $\Phi_n(\mathbf{a}) = (1, -1, 1, -1)$ is $1 \cdot 1 \cdot s_2 \cdot 1 \cdot 1 \cdot s_0s_1 \cdot 1 \cdot s_0 \cdot 1 \cdot 1 = s_2s_0s_1s_0$.

$$\begin{aligned}
(2, 1, -1, 1, -1, -2) &\xrightarrow{s_0} (-1, 1, -1, 1, -1, 1) \xrightarrow{s_1} (1, -1, -1, 1, 1, -1) \xrightarrow{s_3} (1, -1, 1, -1, 1, -1) \\
&\xrightarrow{s_2} (1, 1, -1, 1, -1, -1) \xrightarrow{s_3} (1, 1, 1, -1, -1, -1) \xrightarrow{s_0} (0, 1, 1, -1, -1, 0) \\
&\xrightarrow{s_1} (1, 0, 1, -1, 0, -1) \xrightarrow{s_2} (1, 1, 0, 0, -1, -1) \xrightarrow{s_0} (0, 1, 0, 0, -1, 0) \\
&\xrightarrow{s_1} (1, 0, 0, 0, 0, -1) \xrightarrow{s_0} (0, 0, 0, 0, 0, 0) \\
(1, -1, 1, -1) &\xrightarrow{1} (1, -1, 1, -1) \xrightarrow{1} (1, -1, 1, -1) \xrightarrow{s_2} (1, 1, -1, -1) \\
&\xrightarrow{1} (1, 1, -1, -1) \xrightarrow{1} (1, 1, -1, -1) \xrightarrow{s_0} (0, 1, -1, 0) \\
&\xrightarrow{s_1} (1, 0, 0, -1) \xrightarrow{1} (1, 0, 0, -1) \xrightarrow{s_0} (0, 0, 0, 0) \\
&\xrightarrow{1} (0, 0, 0, 0) \xrightarrow{1} (0, 0, 0, 0)
\end{aligned}$$

Corollary 7.3 (The map Φ_n^k decreases the length by exactly k). *For any abacus \mathbf{a} corresponding to a symmetric core partition in \mathcal{S}_{2n}^k , then $\ell(\Phi_n^k(\mathbf{a})) = \ell(\mathbf{a}) - k$.*

8. PROPERTIES OF THE MAP Φ : BRUHAT ORDER

Fix a hyperplane H_n^k . In this section, we will show that under the identification of H_n^k with \mathbb{R}^{n-1} , the strong Bruhat order is preserved when alcoves with coroot lattice points on H_n^k are projected onto H_n^k . Note that for the rest of this section, x and y are elements of \tilde{C}_n/C_n . By an abuse of notation, we will use the same letters to denote their associated abacus diagrams and core partitions. We begin with a lemma on abacus diagrams.

Lemma 8.1. *Let x and y be elements in \tilde{C}_n/C_n . Then $x \geq_B y$ if and only if for all $k \geq 1$, the k th highest bead in x is as high as the k th highest bead in y .*

Proof. The number of gaps smaller than the k th highest bead in x and y is the length of the k th row in the core partitions of x and y . Let the highest bead in x be located at runner i_x of level ℓ_x , and the highest bead for y at runner i_y of level ℓ_y . By Lemma 5.3, the number of gaps in x that are smaller than the highest bead in x is $2n(\ell_x - 1) + i_x$, and the number of gaps in y that are smaller than the highest bead in y is $2n(\ell_y - 1) + i_y$.

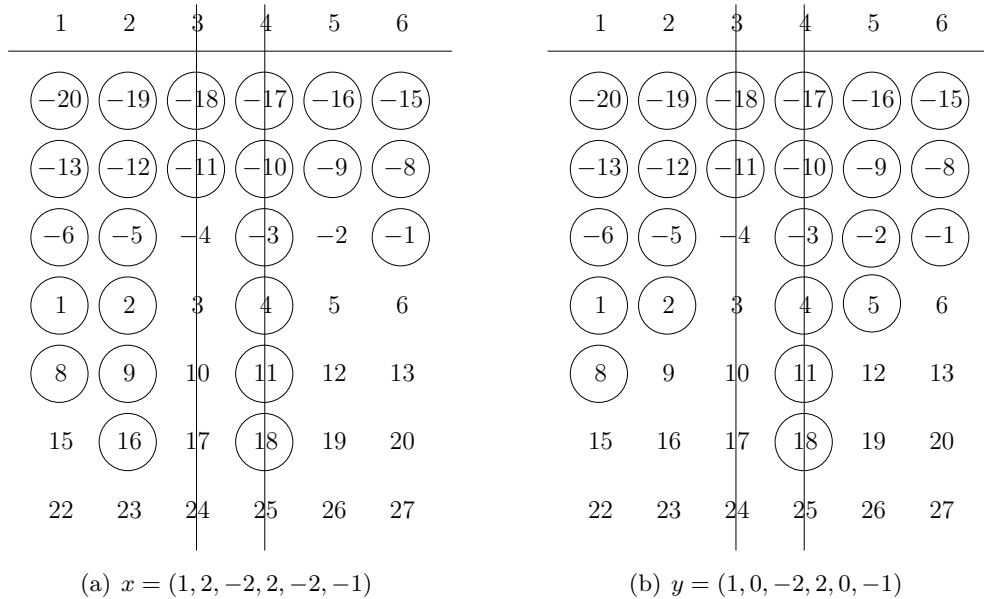
Suppose $x \geq_B y$. It is clear from the previous paragraph that the highest bead in x is as high as the highest bead in y . Moreover, there are $2n(\ell_x - \ell_y) + (i_x - i_y)$ more gaps in x that are smaller than the first highest bead than there are in y . This number is exactly the number of positions that are higher than the highest bead in y but not higher than the highest bead in x . It can be easily checked that since the core partition for x contains the core partition of y , the number of gaps smaller than the k th bead in x is at least the number of gaps smaller than the k th bead in y , and so the k th bead in x must be as high as the k th bead in y . Hence, we have proved the “if” direction.

To prove the converse, note that from the discussion above, if the k th highest bead in x is as high as the k th highest bead in y , then the number of gaps that are smaller than the k th bead in x

Theorem 8.2 (Bruhat order is preserved). *Let x, y be elements in \widehat{C}_n/C_n whose associated coroot lattice points lie on H_n^k . Identify H_n^k with \mathbb{R}^{n-1} and let π be the projection map onto H_n^k . Then $x \geq_B y$ if and only if $\pi(x) \geq_B \pi(y)$.*

Conversely, suppose $\pi(x) \geq_B \pi(y)$. By Lemma 8.1 again, the k th highest bead in $\pi(x)$ is as high as the k th highest bead in $\pi(y)$ for all $k \geq 1$. To obtain x and y from $\pi(x)$ and $\pi(y)$, we are inserting the same beads for both abaci. Given a bead to be inserted, its rank after its insertion to $\pi(x)$ is no higher than its rank after its insertion to $\pi(y)$. Therefore after the addition of all the beads, the k th highest bead in x is still as high as the k th highest bead in y . It follows that $x \geq_B y$, as desired. \square

Example 8.3. For the “if” part, as shown in Figure 12, the beads in $x = (1, 2, -2, 2, -2, -1)$ are numbered **(18, 16, 11, 9, 8, 4, 2, 1, -1, -3, -5, -6, ...)**, and the beads in $y = (1, 0, -2, 2, 0, -1)$ are numbered **(18, 11, 8, 5, 4, 2, 1, -1, -2, -3, -5, -6, -8, ...)**, where the beads to be deleted are bolded. The ranks of the deleted beads in x are $(1, 3, 6, 10, \dots)$, which are no higher than $(1, 2, 5, 10, \dots)$, the ranks of the deleted beads in y . It follows that after the deletion of these beads, the k th highest bead in $\pi(x)$ is still as high as the k th highest bead in $\pi(y)$, so $\pi(x) \geq_B \pi(y)$.



Conversely, for the “only if” part, consider the two abacus diagrams in Figure 13, the beads in $\pi(x) = (1, 2, -2, -1)$ are numbered $(12, 7, 6, 2, 1, -1, -3, -4, \dots)$, and the beads in $\pi(y) = (1, 0, 0, -1)$ are numbered $(6, 3, 2, 1, -1, -2, -3, -4, \dots)$. The ranks of the inserted beads in $\pi(x)$

are $(1, 3, 6, 10, \dots)$, which are no higher than $(1, 2, 5, 10, \dots)$, the ranks of the inserted beads in $\pi(y)$. It follows that after the insertion of these beads, the k th highest bead in x is still as high as the k th highest bead in y (see Figure 12), so $x \geq_B y$.

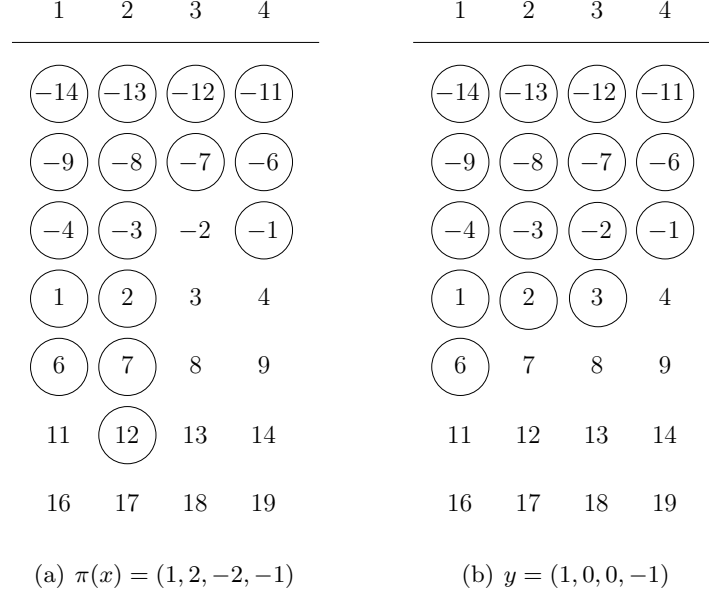


FIGURE 13.

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DEPARTMENT OF MATHEMATICS, HAVERFORD COLLEGE, HAVERFORD, PA
E-mail address: `ebeazley@haverford.edu`

DEPARTMENT OF MATHEMATICS, OBERLIN COLLEGE, OBERLIN, OH
E-mail address: `mnichols@oberlin.edu`

DEPARTMENT OF MATHEMATICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA
E-mail address: `Min.Hae.Park@williams.edu`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA
E-mail address: `dannyshi@mit.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD
E-mail address: `ayoucis@terpmail.umd.edu`